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Four Dimensional Superconformal Field Theories From String Theory

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This thesis is dedicated to my father without whose love
and support it would not have been possible.
It is also dedicated to the memory of my mother.

Abstract

In this work, we discuss four dimensional SCFTs which descend from string theory and M-theory. These SCFTs arise naturally from stacks of string and M- theory branes and are useful tools for studying various phenomena in four dimensional quantum field theory.

In the first chapter we give a brief non-technical introduction and historical background to string theory and supersymmetry which motivates the study of these fields.

The second chapter contains a review of background material related to supersymmetric gauge theories and conformal field theories. In this context, we discuss anomalies and moduli spaces, and their utility in quantum field theory. Additionally, we introduce how to construct some exotic field theories by coupling non-Lagrangian sectors to Lagrangian theories.

In the third chapter, we examine a special family of four dimensional $\mathcal{N} = 2$ SCFTs that are obtained by wrapping the six dimensional $(2, 0)$ theory on a Riemann surface. We derive some properties of these theories and discover various dualities involving them. We then give a brief introduction to the superconformal index and use it to examine the operator content of these theories. Finally we construct new $\mathcal{N} = 1$ gauge theories by coupling these $\mathcal{N} = 2$ theories to $\mathcal{N} = 1$ matter and examining the resulting RG flows.

In the fourth chapter, we discuss the problem of counting chiral primary gauge-invariant operators in D-brane world-volume gauge theories. These theories live on a flat stack of D3-branes with various transverse toric geometries. We investigate the six dimensional transverse space, as well as the moduli space and chiral ring of the world-volume theory, and find previously unknown relationships between these elements.

Acknowledgements

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I would also like to thank the other academics in the Centre for Research in String Theory. They have all always been happy to have a chat about anything physics related or indeed unrelated. Particular thanks go to Sanjaye Ramgoolam with whom I collaborated.

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Finally I would like to thank my family and friends outside of physics for their support. Particular thanks go to my father to whom this thesis is dedicated. His support whether financial, emotional or otherwise throughout the years has led to this thesis.

Declaration

I, James McGrane, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with or supported by others, this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other intellectual property right, or contain any confidential material.

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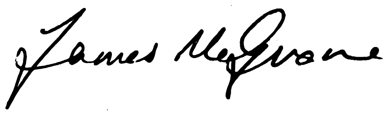

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Details of collaboration and publications:

This thesis describes research carried out in collaboration with Brian Wecht and Sanjaye Ramgoolam. Some of the results of this work have been published in [1] and [2]. Where other sources of information have been used, they have been cited in the bibliography.

Contents

Chapter 1

Introduction

The universe as we know it is described extremely well by two theories of physics. The first of these is the Standard Model (SM) of particle physics which explains physics at extremely small scales, specifically at the subatomic particle level. This theory has been tested and probed by extremely high energy colliders and cosmic observations and has produced some of the most precise measurements in science. The second theory is Einstein's General Theory of Relativity (GR) which describes physics at the largest scales, *e.g.* planetary motion, galactic formation and cosmic evolution. Again this theory has been tested to a high degree of accuracy and seems to be a good description of large scale physics.

There is however a problem with these two theories which is that they are incompatible. In the Standard Model of particle physics we assume that there is no gravity. Hence it works well for describing particle interactions but is useless for describing gravitational interactions. Similarly in General Relativity there is no sign of quantum physics so it is useless for describing subatomic physics. In addition to these problems there are other indications that we do not have a complete description of the universe *e.g.* the existence of dark matter and dark energy, the hierarchy problem and the prediction of black hole space-time singularities none of which are explained by the SM or GR.

For decades physicists have tried to find theories to explain these problems. Two theories which attempt to solve some of the problems mentioned above are Supersymmetry (SUSY) and String Theory which form the cornerstones of this thesis.

Fundamental particles in our universe very generally come in two types, bosons and fermions. Supersymmetry states that all particles come in pairs and that for every boson there is a fermion and for every fermion there is a boson. It has been suggested that Supersymmetry if broken at a low enough energy could partially solve the hierarchy problem and could also contain dark matter candidates.

Modern theories of particle physics state that all particles are point-like; that is, no matter how far we zoom in on a particle the particle still just looks like a point in space-time. In String Theory it is postulated that if we zoom in far enough we will see that particles are actually extended in one dimension and appear like a piece of string. The strings can vibrate and different frequencies of vibration correspond to different particles. The reason that String Theory is interesting is that it is a consistent theory of quantum gravity and particles physics. That is to say at small length scales it looks like quantum physics and at large length scales it looks like gravity. It is for this reason that people originally studying String Theory thought that it might be a grand unifying theory of everything for physics in our universe.

It should be stated however that obtaining a version of String Theory that describes our universe is non-trivial and it is in fact not obvious that String Theory can describe our universe at all. String Theory has been criticised for the fact that it can accommodate a large number (possibly infinite number) of vacua only one of which would describe our universe. Furthermore String Theory contains massless bosons which have not been observed in our universe. For these reasons amongst others hopes of String Theory being a unifying theory of everything have been somewhat diminished over the years.

Nonetheless, String Theory remains a fascinating field of study precisely because it is a consistent theory of quantum gravity. String Theory also contains some complex mathematics and has led to many interesting discoveries in mathematics. It may also be the case that learning to understand String Theory gives us a deeper understanding of mathematical physics and potentially leads us to a theory which does describe our universe.

As well as containing one dimensional strings, String Theory also contains higher dimensional objects called "branes" which will play an important role in this thesis. Strings can begin and end on these branes and from the point of view of branes which see the end-points of the strings these end-points describe particle physics interactions. Thus, if we lived on the world-volume of one of these branes we would only see the end-points of the strings and we would see a theory of particle physics.

Another peculiarity of String Theory is that it is only mathematically consistent in ten space-time dimensions. One way to obtain a four dimensional theory from String Theory is to postulate that the extra six dimensions are wrapped up and very small. Another way to obtain a four dimensional theory is to look at the world-volume of a three dimensional brane and examine the physics that lives on the world-volume arising from strings interacting with/on the brane. Finally a third way to obtain a four dimensional theory from String Theory is a combination of these two: take the world-volume theory that lives on a $(3+k)$ dimensional brane and compactify k dimensions. In this thesis we will deal with theories that live on the world-volume of branes both four dimensional and higher dimensional which need to be compactified. Ten dimensional String Theory can also be obtained by taking an eleven dimensional theory, M-theory, and compactifiying on a circle.

The goal of the current work is to examine some of the four dimensional quantum field theories that we can obtain from String Theory. In particular the four dimensional field theories that are studied in this thesis arise from studying brane world-volume gauge theories in String Theory and M-theory. By studying these theories we hope to gain greater insight into mathematical physics that might be useful in describing the physics of our universe.

The four dimensional quantum field theories that we study in this thesis are typically much more exotic than the physics of our universe. For example, they all contain some level of Supersymmetry and normally do not contain massive fields. While these aspects might seem undesirable at first since we do not observe them in our universe it is often the case that these aspects allow us to examine the theory in greater detail and calculate quantities that would otherwise be difficult/impossible to calculate.

As for the structure of this thesis, in chapter 2 we present the basics of SUSY gauge theories, Conformal Field Theories (CFTs) and Superconformal Field Theories (SCFTs), knowledge of which is necessary to understand the research that is presented in later chapters. We talk about quiver theories, non-Lagrangian field theories and generalised quiver theories. We also talk about moduli spaces and describe a few simple examples which will be helpful for understanding more complex examples in later chapters. We finish with a discussion of conformal manifolds which arise naturally in some conformal field theories that we study in later chapters.

In chapter 3 we take a stack of five dimensional branes in M-theory and compactify two of the world-volume dimensions. The resulting so-called theories of class $\mathcal{S}$ are four dimensional $\mathcal{N} = 2$ superconformal field theories. We present a sub-family of theories of class $\mathcal{S}$ and examine dualities that they appear in. We also use the superconformal index to study the operator content and examine flows that arise when we couple these theories to $\mathcal{N} = 1$ matter.

In chapter 4 we look at the problem of counting chiral primary gauge-invariant operators in D-brane world-volume gauge theories. In this setup we take a flat stack of D3 branes in a ten dimensional space where the space transverse to the branes can be some non-trivial space; this non-trivial transverse space gives rise to the field content of the world volume gauge theory. We look at various different transverse geometries and find some interesting relationships between the transverse geometry, the moduli space of the world-volume gauge theory and the chiral ring of the world-volume gauge theory.

Chapter 2

Review

In this chapter we review the basic technology of Conformal Field Theories (CFTs), $\mathcal{N} = 1, 2, 4$ supersymmetry (SUSY) and Superconformal Field Theories (SCFTs). This chapter is meant to introduce some basic topics which will be taken for granted in later chapters. More advanced topics will be introduced as necessary in later chapters.

2.1 Conformal Field Theories

Conformal field theories are field theories that are invariant under conformal transformations. That is to say they are invariant under a co-ordinate transformation, $x \to x'$, that leaves the metric invariant up to a position-dependent scale change:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x), \tag{2.1}$$

where $g_{\mu\nu}$ is the metric in flat space-time and x^μ are the space-time co-ordinates. These transformations preserve angles between vectors and contain as a subgroup the group of Lorentz transformations (corresponding to $\Omega(x) = 1$). The group of conformal transformations also contains as a subgroup the group of scale transformations, $x^\mu \to x'^\mu = \lambda x^\mu$, (corresponding to $\Omega(x) = \lambda^{-2}$). Because of this any field theory that has an associated length scale is not scale invariant and thus not conformally invariant. Infinitesimal conformal transformations are transformations $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$ where there is some constraint on ϵ^μ which we now derive. The line element transforms as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \to g'_{\mu\nu} dx'^\mu dx'^\nu = dx^\mu dx^\nu \left(g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right). \tag{2.2}$$

For this to be a conformal transformation we need that $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = C(x) g_{\mu\nu}$. Multiplying both sides of this equation by $g_{\mu\nu}$ gives us that $C(x) = \frac{2}{d} \partial \cdot \epsilon$. This gives us the equation that constrains the ϵ^μs for a conformal transformation,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \left(\partial \cdot \epsilon\right) g_{\mu\nu}. \tag{2.3}$$

After multiplying both sides by $\partial_\rho \partial^\nu$ (summing over ν) and interchanging $\rho \leftrightarrow \mu$ we get

$$\left(g_{\mu\nu}\partial^2 + (d-2)\,\partial_\mu\partial_\nu\right)\partial \cdot \epsilon = 0. \tag{2.4}$$

When we specialise to $d = 2$ equation (2.3) tells us that

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \qquad\qquad \partial_1\epsilon_2 = -\partial_2\epsilon_1. \tag{2.5}$$

If we define the complex co-ordinate $\epsilon(z) = \epsilon_1(z) + i\epsilon_2(z)$, where $z = x^1 + ix^2$ then these constraints are just the Cauchy-Riemann equations. Thus ϵ can be any analytic function of z. In terms of finite transformations this means that the theory is invariant under the transformations

$$z \to f(z), \qquad\qquad \overline{z} \to \overline{f}(\overline{z}), \tag{2.6}$$

the local algebra of which is infinite dimensional. For $d \neq 2$ equation (2.4) tells us that ϵ can be at most quadratic in the co-ordinates and thus the algebra is finite dimensional.

A consequence of conformal invariance is that the expectation value of the trace of the energy-momentum tensor vanishes. It is not difficult to see the converse classically. To see this we note that under an infinitesimal conformal transformation the variation in the action is

$$\delta S = \int d^d x T^{\mu\nu}\partial_\mu\epsilon_\nu = \frac{1}{d}\int d^d x \partial_\rho \epsilon^\rho T^\mu{}_\mu. \tag{2.7}$$

So if the trace of the energy momentum tensor vanishes the variation of the action vanishes and so the theory is conformally invariant.

Because a conformal field theory is also scale invariant the dilatation current

$$j_\nu = x^\mu T_{\mu\nu} \tag{2.8}$$

is conserved. Since the energy-momentum tensor is conserved in a Poincaré invariant theory the conservation equation $\partial^\mu j_\mu = 0$ implies that the trace of the energy momentum tensor vanishes in a conformal field theory.

Quantum mechanically the corresponding Ward identity [3] tells us expectation value of the trace of the energy-momentum tensor vanishes $\langle T^\mu{}_\mu \rangle = 0$. There is a caveat here though and this is that the trace of the energy-momentum tensor will only vanish in flat space-time. This is because introducing curvature means introducing a length scale and so in non-flat space-time the trace of the energy-momentum tensor generally does not vanish. This is known as the Weyl anomaly and in quantum theories that are conformally invariant in flat space-time the trace of the energy-momentum tensor has a prescribed structure when the theory is put on a non-flat background. In 2 dimensions it is [3]

$$\left\langle T^\mu{}_\mu \right\rangle = \frac{c}{24\pi} R, \tag{2.9}$$

where R is the Ricci scalar and in 4 dimensions it is [4]

$$\langle T^{\mu}{}_{\mu} \rangle = \frac{c}{16\pi^2} \, (\mathrm{Weyl})^2 - \frac{a}{16\pi^2} \, (\mathrm{Euler}) \, , \tag{2.10}$$

where

$$(\mathrm{Weyl})^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2, \qquad (\mathrm{Euler}) = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2. \tag{2.11}$$

CFTs are characterised by the anomaly coefficients a,c (only c for $d = 2$) that appear in the trace.

Because the group of conformal transformations in 2 dimensions is infinite dimensional it is easier to find examples of interacting 2d conformal field theories. This is because there are more constraints on the theory. Interacting CFTs in 4 dimensions (and also any other number of dimensions not equal to 2) however are much harder to find than 2 dimensional ones. The first example of an interacting 4 dimensional conformal field theory was $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) [5].

$\mathcal{N} = 4$ SYM can be thought of as an $\mathcal{N} = 1$ SUSY gauge theory with 3 chiral supermultiplets that transform in the adjoint representation and a particular choice of superpotential or instead as an $\mathcal{N} = 2$ gauge theory with one adjoint hypermultiplet. The $\mathcal{N} = 4$ theory contains the right matter content in the single $\mathcal{N} = 4$ multiplet and the right symmetries to ensure that the theory is invariant under conformal transformations. Because the theory is conformally invariant it is necessarily scale invariant and in particular this means that the theory does not flow because there is no energy scale dependence.

It is important to emphasise that it is in general very difficult to find an interacting quantum field theory that is conformally invariant. Renormalising a theory involves introducing an energy scale and this energy scale breaks conformal invariance except at certain values of the parameters of the theory which constitute a renormalisation group fixed point. For the $\mathcal{N} = 4$ SYM theory, we presented the structure of the theory and said that there is no RG flow however we could have had the $\mathcal{N} = 4$ theory as a conformal fixed point in an RG flow. These "conformal fixed points" exist at asymptotically high energies, "UV fixed points", or asymptotically low energies, "IR fixed points", and we normally talk about flowing from the UV to the IR.

In gauge theories that are conformally invariant one consequence of conformal invariance is the vanishing of the beta function for the gauge coupling constant ($\beta = \mu\frac{\partial g}{\partial \mu}$, where g is the gauge coupling constant and μ is the energy scale). A non-vanishing beta function would imply an energy scale dependence and thus break scale invariance which is a subset of conformal invariance.

The beta function for the gauge coupling constant of a theory with a UV fixed point is plotted on the left of figure 2.1. When the gauge coupling constant satisfies $g < g^*$ we have that $\beta = \frac{dg}{d\mu} > 0$ (where μ is the energy scale) and when $g > g^*$ we have $\beta < 0$. This means that as the energy scale tends to infinity the gauge coupling constant g tends to g^*.

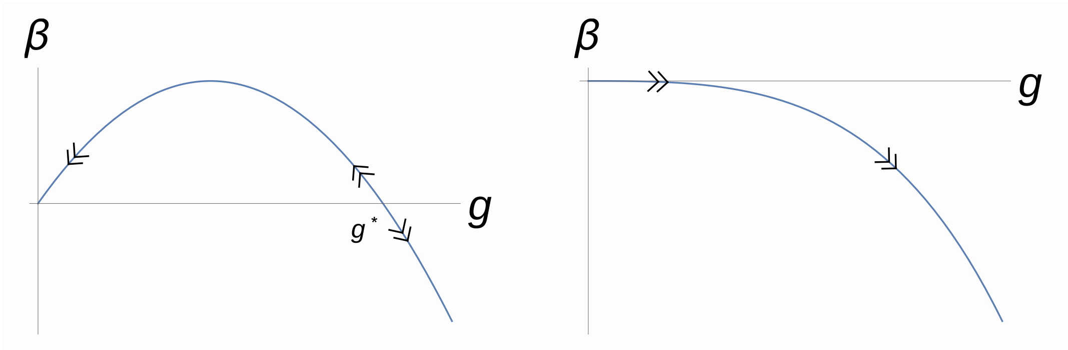

Figure 2.1: Left: Sketch of the beta function for a theory with a UV fixed point. Right: Sketch of the beta function for an asymptotically free theory near $g = 0$. Arrows indicate the direction of RG flow from the UV to the IR.

On the right of figure 2.1 the beta function for an asymptotically free theory is sketched near $g = 0$; at higher g the beta function can decrease monotonically or reach a minimum. Because β is less than zero the gauge coupling constant g tends to 0 as the energy scale tends to infinity.

The first evidence to support the idea that CFTs existed as the IR phases of quantum field theories came in the form of the Banks-Zaks fixed point [6, 7]. The Banks-Zaks fixed point is an example of a weakly coupled conformal fixed point in an asymptotically free theory. The beta function for the gauge coupling constant, g, for an $SU(N)$ gauge theory with N_F Dirac fermions in the fundamental representation of $SU(N)$ is

$$\beta(g) = \mu \frac{dg}{d\mu} = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} + \mathcal{O}\left(g^7\right), \tag{2.12}$$

where

$$\begin{aligned} \beta_0 &= \frac{1}{3}\left(11N_C - 2N_F\right), \\ \beta_1 &= \frac{34}{3}N_C^2 - \frac{1}{2}N_F\left(2\frac{N_C^2-1}{N_C} + \frac{20}{3}N_C\right), \end{aligned}$$

and where μ is the energy scale. When $N_F < \frac{11}{2}N_C$, β_0 is positive and the beta function is negative close to $g = 0$, i.e. this is the condition on N_F for asymptotic freedom. The beta function looks like the one sketched on the right of figure 2.1 close to $g = 0$. From equation (2.12) we can see that there is a also zero of the beta function at $g = g^*$ where $g^{*2} = -\frac{\beta_0(16\pi^2)}{\beta_1}$. Thus there will be a second zero of the beta function when β_1 is negative. This occurs for values of N_F that are close to but less than $\frac{11}{2}N_C$. This scenario is sketched in figure 2.2. If we let $N_F = \frac{11}{2}N_C - n$ where n is a small positive half-integer such that β_1 is negative (and such that n is of order $\mathcal{O}\left(N_C^0\right)$) then the zero of the beta function, g^*, will be of order $\frac{1}{N_C}$ and the 't Hooft coupling parameter $g^{*2}N_C$ will be of order $\frac{1}{N_C}$. This means that if we take N_C large enough the conformal fixed point will occur at a value of g where perturbative methods are still valid.

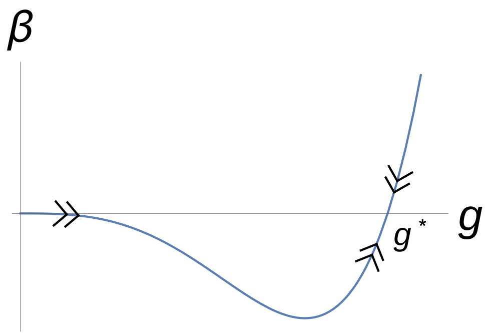

Figure 2.2: Beta function for an asymptotically free gauge theory with an IR fixed point.

Another example of a 4d CFT came with the discovery of Seiberg duality [8]. This deals with the theory of Supersymmetric Quantum ChromoDynamics (SQCD). This is an $\mathcal{N} = 1$ supersymmetric theory of N_F chiral superfields that transform in the fundamental representation of an $SU(N_C)$ gauge group (quarks) and N_F chiral superfields that transform in the anti-fundamental representation of $SU(N_C)$ (anti-quarks). This theory is asymptotically free when the number of flavours is less than 3 times the number of colours, i.e. $N_F < 3N_C$. The charges under global symmetries and the gauge symmetry are tabulated below. The $U(1)_R$ charges are determined by ensuring that the symmetry

	$SU(N_C)$	$SU(N_F)$	$SU(N_F)$	$U(1)_B$	$U(1)_R$
Q	$\square$	$\square$	$\mathbf{1}$	1	$\frac{N_F - N_C}{N_F}$
$\widetilde{Q}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	-1	$\frac{N_F - N_C}{N_F}$

Table 2.1: Charges of the quarks and anti-quarks under gauge and global symmetries.

is anomaly-free (more about this later). It is a general result of SFCTs that

$$D[\mathcal{O}] \geq \left| \frac{3}{2} R[\mathcal{O}] \right| , \tag{2.13}$$

where $D[\mathcal{O}]$ and $R[\mathcal{O}]$ are the scaling dimension and R charge of a gauge invariant operator $\mathcal{O}$ respectively. The inequality is saturated for chiral/anti-chiral primary operators. In SQCD the mesons, $M = Q\widetilde{Q}$, are gauge invariant chiral primaries. The Q superfields have scaling dimension, $D[Q] = 1 + \frac{1}{2}\gamma$, where γ is the anomalous dimension of the Q superfields. The anomalous dimension is 0 in the limit where the quarks are free fields and then varies away from the free field limit. Thus in SQCD if the theory tends to a conformal fixed point on the RG flow the inequality in eq (2.13) will be satisfied at that point. The R charge in eq (2.13) is the superconformal R charge. Since there is only one anomaly-free R-charge that preserves $Q \to \widetilde{Q}$ invariance (the one given in table 2.1), this R-charge can be identified as the superconformal R-charge if the theory flows to a fixed point. In [8] it is argued that $\mathcal{N} = 1$ SQCD tends to an SCFT and that there is a dual theory that also tends to the same SCFT. At the conformal fixed point the mesons obey

$$\begin{aligned} D[M] &= \frac{3}{2}R[M], \\ \Rightarrow 2+\gamma &= \frac{3}{2}2\frac{N_F-N_C}{N_F}, \\ \Rightarrow \gamma &= 1-\frac{3N_C}{N_F}. \end{aligned} \tag{2.14}$$

This tells us the value of the anomalous dimension γ of the Q superfield at the conformal fixed point.

From the beta function for the gauge coupling constant which will be given later (equation (2.22)) one can see that the beta function for SQCD obeys

$$\beta(g) \propto \left(3N_C - N_F\left(1-\gamma\right)\right), \tag{2.15}$$

and by substituting in the value of γ given in equation (2.14) it can be seen that at this point on the RG flow the beta function vanishes as is required of the beta functions for any SCFT.

The theories mentioned above along with many others have lead to the realisation there is a rich family of conformal field theories in 4 dimensions. The ones that we will be interested in in this thesis are superconformal field theories (SCFTs) such as the last example and $\mathcal{N}=4$ SYM.

2.2 Anomalies

In the last section we used an anomaly constraint to determine the anomaly-free $U(1)_R$ symmetry and in later chapters we will use anomaly constraints to determine when a theory is consistent and when it is not. Here we briefly sketch what anomalies are and how to determine if they spoil a theory or not. This section roughly follows the introduction given in [9].

An anomaly refers to a symmetry which is preserved in a classical theory but is broken by quantum effects. If we have some symmetry classically then it has an associated current:

$$j^\mu_a = \overline{\psi}^{\dot{\alpha}}_j \overline{\sigma}^\mu_{\alpha\dot{\alpha}} (T_a)^j_i \psi^{i\alpha}, \tag{2.16}$$

where T_a are the generators of the symmetry group. For this to be a symmetry classically means that $\partial_\mu j^\mu_a = 0$. However, even if the symmetry is preserved classically the symmetry can be broken by "triangle" diagrams such as those shown in figure 2.3. These triangle diagrams contribute to the three-point functions of currents and so the divergence of a current will be of the form

$$\partial_\mu j^\mu_a \propto \mathrm{Tr}(T_a T_b T_c) F^b_{\mu\nu} F^c_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}. \tag{2.17}$$

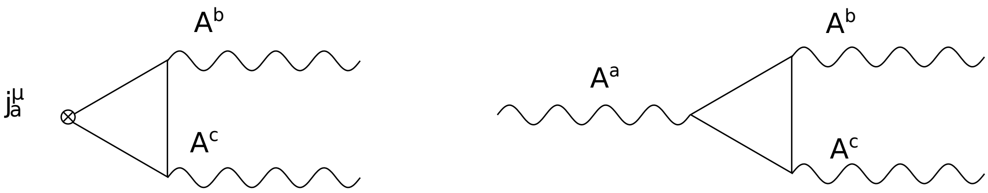

Figure 2.3: Triangle diagrams for: 2 gauge currents and 1 global current (left); three gauge currents (right)

For this to vanish we require that $\sum_i \mathrm{Tr}_{\mathbf{r}_i}(T_a\{T_b, T_c\}) = 0$, where the sum is over all fermions, in representations $\mathbf{r}_i$, that can run in the loop of these triangle diagrams.

If the current we are computing the anomaly for is a global current, then if we calculate all triangle diagrams with two gauge fields in them such as that shown on the left of figure 2.3 then if the sum does not vanish this means that the global symmetry is not preserved by quantum effects.

If the current we are computing the anomaly for is a gauge current, then we must add all diagrams which are like the one on the right of figure 2.3. If the sum of these does not vanish then this is much more serious as it implies that the theory is inconsistent.

In this thesis we will often be interested in calculating the anomaly for $U(1)$ global symmetries. When we are calculating the gauge-gauge-global anomaly the anomaly splits up into

$$\mathcal{A} = \mathrm{Tr}(T_a T_b T_A) = \mathrm{Tr}(T_a T_b)\,\mathrm{Tr}(T_A), \tag{2.18}$$

where T_a are the generators of the gauge symmetry and T_A are the generators of the global symmetry. When the global symmetry is a $U(1)$ symmetry this becomes

$$\mathcal{A} = \sum_i J_i T(\mathbf{r}_i), \tag{2.19}$$

where the sum is over fermions that can run in the loop, J_i is the $U(1)$ charge of the fermion and $\mathbf{r}_i$ is the representation of the gauge group that the fermion is in. $T(\mathbf{r})$ is the index of the representation, $\mathbf{r}$, defined by $T(\mathbf{r})\delta_{ab} = \mathrm{Tr}(T_a^{\mathbf{r}} T_b^{\mathbf{r}})$.

As an example we will now compute the anomaly-free $U(1)_R$ charges for $\mathcal{N} = 1$ SQCD that were quoted without derivation in table 2.1. In order to preserve the classical $Q \leftrightarrow \widetilde{Q}$ symmetry we must have that $R(Q) = R(\widetilde{Q}) = R$. Then the anomaly of the $U(1)_R$ current is $\sum \mathrm{Tr}(SU(N)^2 U(1)_R)$ which gives us

$$1 \cdot T(\mathbf{Ad}) + (R-1)T(\square)2N_F = 0, \tag{2.20}$$

where $T(\mathbf{r}_i)$ is the index of the representation and $\square$, $\overline{\square}$ and $\mathbf{Ad}$ mean the fundamental, anti-fundamental and adjoint representations respectively. For $SU(N)$ we can normalise the generators so that $T(\square) = T(\overline{\square}) = 1/2$ and $T(\mathbf{Ad}) = N$. A different normalisation will rescale these quantities by the same factor so the ratio between these two never changes.

This gives us

$$R = \frac{N_F - N_C}{N_F}. \tag{2.21}$$

This is the value of the R-charge that was stated in table 2.1.

2.3 $\mathcal{N} = 1$, $\mathcal{N} = 2$, Quivers and SCFTs

Gauge theories with $\mathcal{N} = 1$ supersymmetry are constructed from chiral and vector multiplets. A chiral multiplet contains a Weyl fermion and a scalar field and a vector multiplet contains a Weyl fermion and a vector field:

$$\text{Chiral Multiplet}: \begin{pmatrix} Q \\ \psi \end{pmatrix}, \qquad \text{Vector Multiplet}: \begin{pmatrix} A_\mu \\ \lambda \end{pmatrix}.$$

We can use these representations to build $\mathcal{N} = 1$ quiver theories. An $\mathcal{N} = 1$ quiver theory is any theory that can be described by an $\mathcal{N} = 1$ quiver diagram such as that given in figure 2.4. An $\mathcal{N} = 1$ quiver diagram is made using three components: boxes, circles and

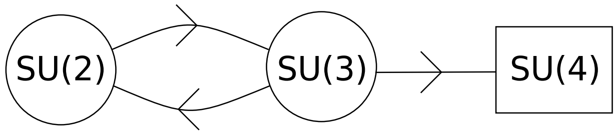

Figure 2.4: Quiver diagram for an $\mathcal{N} = 1$ $SU(2) \times SU(3)$ gauge theory with 1 chiral multiplet in the $(\mathbf{2}, \overline{\mathbf{3}})$ representation, 1 chiral multiplet in the $(\overline{\mathbf{2}}, \mathbf{3})$ and 4 chiral multiplets in the $(\mathbf{1}, \mathbf{3})$ representation.

directed lines. A circle denotes a gauge group and thus an $\mathcal{N} = 1$ vector multiplet; a directed line denotes a chiral multiplet; and a box denotes a flavour symmetry. A directed line going from one circle (box) to another circle (box) denotes a chiral multiplet in the $\square$ representation of the first gauge (flavour) group and in the $\overline{\square}$ of the second gauge (flavour) group.

When we move to $\mathcal{N} = 2$ theories the representations needed for constructing gauge theories are the $\mathcal{N} = 2$ vector multiplet and the $\mathcal{N} = 2$ hypermultiplet. The $\mathcal{N} = 2$ hypermultiplet consists of two $\mathcal{N} = 1$ chiral multiplets (i.e. two Weyl fermions and two scalars) and the $\mathcal{N} = 2$ vector multiplet consists of one $\mathcal{N} = 1$ vector multiplet and one $\mathcal{N} = 1$ chiral multiplet (i.e. two Weyl fermions, one scalar and one vector):

$$\text{Hypermultiplet}: \begin{pmatrix} & \psi & \\ Q & & \widetilde{Q}^\dagger \\ & \widetilde{\psi}^\dagger & \end{pmatrix}, \text{Vector Multiplet}: \begin{pmatrix} & A_\mu & \\ \lambda & & \lambda' \\ & \phi & \end{pmatrix}.$$

We can use these representations to build $\mathcal{N} = 2$ quiver theories, where an $\mathcal{N} = 2$ quiver theory is one that can be described using an $\mathcal{N} = 2$ quiver diagram such as the one in figure 2.5. $\mathcal{N} = 2$ quiver diagrams like their $\mathcal{N} = 1$ counterparts consist of boxes, circles and lines but now the circles denote $\mathcal{N}=2$ vector multiplets and lines denote $\mathcal{N}=2$ hypermultiplets. The lines are no longer directed because every hypermultiplet contains two $\mathcal{N}=1$ chiral multiplets in conjugate representations. We call these hypermultiplets, with one chiral multiplet in the $(\square,\overline{\square})$ and one chiral multiplet in the $(\overline{\square},\square)$, "bifundamental" hypermultiplets.

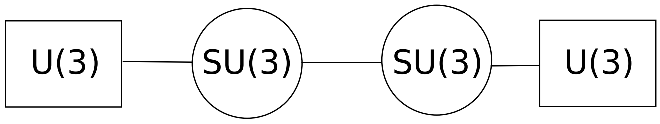

Figure 2.5: $\mathcal{N}=2$ quiver diagram for an $SU(3)^2$ gauge theory with 1 bifundamental hypermultiplet, three hypermultiplets that transform in the fundamental representation of $SU(3)_1$ and three hypermultiplets that transform in the fundamental representation of $SU(3)_2$

For the quiver shown in figure 2.5, the gauge group is $SU(3)_1 \times SU(3)_2$. There is a bifundamental hypermultiplet which consists of two $\mathcal{N}=1$ chiral multiplets: one $\mathcal{N}=1$ chiral multiplet, Q, transforms in the $(\square,\overline{\square})$ representation of $SU(3)_1 \times SU(3)_2$ and by $\mathcal{N}=2$ symmetry the second chiral multiplet, $\widetilde{Q}$, transforms in the $(\overline{\square},\square)$ representation of $SU(3)_1 \times SU(3)_2$. There are also 3 hypermultiplets that transform in the fundamental representation of $SU(3)_1$. These have an $U(3)$ flavour symmetry which is represented by the left-most box in the quiver. Likewise for $SU(3)_2$.

Many of the theories that are constructed later consist of strongly interacting superconformal field theories. As discussed in the previous section, a necessary condition for conformality is the vanishing of the beta function. For $\mathcal{N}=1$ theories the exact beta function is given by [10]

$$\beta_{8\pi^2/g^2} = \frac{\partial}{\partial \log \mu}\frac{8\pi^2}{g^2} = \frac{3T(\mathbf{Adj}) - \sum_i T(\mathbf{r}_i)(1-\gamma_i(g))}{1 - \frac{g^2 T(\mathbf{Adj})}{8\pi^2}}, \tag{2.22}$$

where μ is the energy scale, g is the gauge coupling constant and γ_i is the anomalous dimension of the i-th chiral superfield. The sum is over all chiral superfields and $T(\mathbf{r}_i)$ is the index of the representation $\mathbf{r}_i$ of the gauge group that the i-th chiral multiplet is in.

For $\mathcal{N}=2$ theories, because there are no anomalous dimensions[1], the condition for conformality is[2]

$$3T(\mathbf{Adj}) - \sum_i T(\mathbf{r}_i) = 0. \tag{2.23}$$

We can construct a simple $\mathcal{N}=2$ supersymmetric gauge theory by taking a vector multiplet and N_F hypermultiplets that transform in the fundamental representation of the gauge group. This theory is called $\mathcal{N}=2$ SQCD. If we do this then the condition for

[1]This is true for $\mathcal{N}=2$ theories with marginal gauge couplings. There are theories however, *e.g.* Argyres-Douglas theories [11] where the holomorphic gauge coupling, τ, is fixed at some value and in these theories there are anomalous dimensions.

[2]This equation is written in $\mathcal{N}=1$ language so the sum is over all $\mathcal{N}=1$ chiral superfields, including the adjoint chiral superfield that lives in the $\mathcal{N}=2$ vector multiplet.

conformality becomes

$$3T(\mathbf{Adj}) - T(\mathbf{Adj}) - 2N_F T(\Box) = 0. \tag{2.24}$$

The first term in the formula above comes from the $\mathcal{N} = 1$ vector superfield. The second term comes from the $\mathcal{N} = 1$ adjoint chiral superfield which lives inside the $\mathcal{N} = 2$ vector multiplet. The last term is for the $2N_F$ $\mathcal{N} = 1$ chiral superfields; there are 2 $\mathcal{N} = 1$ chiral superfields for each hypermultiplet, one in the $\Box$ representation and one in the $\overline{\Box}$ representation but since $T(\Box) = T(\overline{\Box})$ we get the final term.

If the gauge group is $SU(N)$ then $T(\Box) = \frac{1}{2}$ and $T(\mathbf{Adj}) = N$ are the indices for the fundamental and adjoint representation. This means that the condition on the number of hypermultiplets that we should have for superconformality in $SU(N)$ $\mathcal{N} = 2$ SQCD is

$$N_F = 2N. \tag{2.25}$$

One final aspect which should be mentioned is what happens when we give a vev to a scalar field in an $\mathcal{N} = 2$ theory. Two ways of doing this are often referred to as moving out on the Coulomb branch and moving out on the Higgs branch. The Coulomb branch is the branch we move onto when we give vev's to the scalar fields in $\mathcal{N} = 2$ vector multiplets (i.e. $\langle \phi \rangle \neq 0$). It is so-called because the $SU(N)$ gauge symmetry is then generically broken to $U(1)^{N-1}$. The Higgs branch is the branch we move onto when we give vev's to the scalars in hypermultiplets (i.e. $\left\langle Q\widetilde{Q} \right\rangle \neq 0$). We could of course give vev's to the scalars in vector multiplets and hypermultiplets provided the D- and F-terms are satisfied; this is referred to as moving out on the mixed branch. The reason that we highlight the Coulomb and Higgs branches is because there are a lot of interesting results that can be obtained for them.

2.4 Generalised Quiver Theories

In chapter 3 we will be interested in constructing theories by coupling non-Lagrangian theories to Lagrangian theories. In this section we introduce some of the necessary background information on how exactly this is done.

A non-Lagrangian theory is any theory for which we do not have a construction in terms of a Lagrangian. Instead we normally have a construction in terms of compactification of a higher dimensional theory such as the 6 dimensional $(2, 0)$ theory. There may indeed be a construction for these "non-Lagrangian theories" in terms of a Lagrangian however the quality of being non-Lagrangian means that no such construction is known. These theories in general have some flavour symmetry which we can gauge and it is by gauging these flavour symmetries of non-Lagrangian theories that we couple non-Lagrangian and Lagrangian sectors together.

In order to construct these theories where non-Lagrangian and Lagrangian theories are coupled together we must first answer the question of what happens when we gauge the flavour symmetry (or some subset of the flavour symmetry) of a non-Lagrangian SCFT. Once we have done this we will look at constructing theories.

2.4.1 Central Charge of the Flavour Symmetry

This section follows the nice introductions given in [4, 12] about coupling Lagrangian and non-Lagrangian theories.

Any superconformal field theory with a flavour symmetry has a real dimension 2 operator $\mathcal{J}^a$ that transforms in the adjoint representation of the flavour symmetry. This operator is the current superfield and contains the conserved current j^a_μ in the $\theta\sigma^\mu\bar{\theta}$ component.

Gauging the flavour symmetry, or some subset thereof, of some non-Lagrangian SCFT just entails adding a term to the Lagrangian of the form $\int d^4\theta \mathcal{J}^a\mathcal{V}^a$, where $\mathcal{V}^a$ is a vector superfield. This means that we will have some non-Lagrangian sector, some Lagrangian sector and an extra interaction term in the Lagrangian, $\mathcal{J}^a\mathcal{V}^a$, which which couples the two sectors. This is precisely the same prescription we use when gauging a flavour symmetry of a set of hypermultiplets. In this case the flavour symmetry has the current superfield $\mathcal{J}^a = \Phi^\dagger T^a \Phi$ (where T^a are the generators of the flavour symmetry group) which we gauge in the same way as just mentioned.

The contribution of the non-Lagrangian sector to the beta function of the coupling constant for the gauge group is proportional to the central charge of the flavour symmetry, k_G. The central charge of the flavour symmetry is defined via the flavour symmetry OPE [12]:

$$j^a_\mu(x)j^b_\nu(0) = \frac{3k_G}{4\pi^4}\delta^{ab}\frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{x^8} + \frac{2}{\pi^2}f^{abc}\frac{x_\mu x_\nu x\cdot j^c(0)}{x^6} + \cdots, \tag{2.26}$$

where f^{abc} are the structure constants for the flavour symmetry group.

n free hypermultiplets have a $U(n)$ flavour symmetry and the central charge of this flavour symmetry is $k_{U(n)} = 1$; the central charge of the flavour symmetry of some subgroup $G \subset U(n)$ is given by

$$k_G = 2\sum_i T(r_i), \tag{2.27}$$

where the fundamental representation of $U(n)$ decomposes as $\mathbf{n} = \oplus_i \mathbf{r}_i$.

If there is a flavour symmetry H with a gauged subgroup G where the fundamental representation of H, $\mathbf{r}$, decomposes as $\mathbf{r} = \oplus_i \mathbf{r}_i$ then the contribution to the beta function is proportional to $k_{G\subset H}$ where

$$k_{G\subset H} = I_{G\hookrightarrow H} k_H, \tag{2.28}$$

and $I_{G\hookrightarrow H}$ is the embedding index:

$$I_{G\hookrightarrow H} = \frac{\sum_i T(\mathbf{r}_i)}{T(\mathbf{r})}. \tag{2.29}$$

Now we look at how to generalise the beta function to superconformal field theories with a non-Lagrangian sector. If we use holomorphically normalised gauge fields [13] then the denominator in equation (2.22) is identically 1. We can rewrite the numerator as[3]

$$3T(\mathbf{Adj}) - \sum_i T(\mathbf{r}_i)(1-\gamma_i) = 3T(\mathbf{Adj}) + 3\sum_i (R_i - 1)T(\mathbf{r}_i) = X, \tag{2.30}$$

where $X\delta^{ab} = 3\,\mathrm{Tr}\,RT^aT^b$, R_i is the R-charge of the superfields and the trace is over the Weyl fermions (the gauginos have been incorporated into the trace since they have R-charge 1). The R symmetry in this formula is the exact superconformal R symmetry. Also the final term contains an extra δ^{ab} factor.

At one loop (obtained by setting $\gamma_i = 0$) the beta function for a theory with a Lagrangian sector and a non-Lagrangian sector is

$$\beta_{\text{1-loop}} = \underbrace{3T(\mathbf{Adj})}_{\substack{\mathcal{N}=1 \text{ vector} \\ \text{multiplets}}} - \underbrace{\sum_i T(\mathbf{r}_i)}_{\substack{\mathcal{N}=1 \text{ matter} \\ \text{multiplets}}} - \underbrace{\frac{k_G}{2}}_{\substack{\text{Non-Lag} \\ \text{sector}}}, \tag{2.31}$$

where we have used equation (2.27) to obtain the 1-loop contribution of the non-Lagrangian sector.

Using

$$R[\mathcal{O}] = \frac{2}{3}D[\mathcal{O}] = \frac{2}{3}\left(D_{UV}[\mathcal{O}] + \frac{1}{2}\gamma[\mathcal{O}]\right), \tag{2.32}$$

the full beta function is

$$\beta = 3T(\mathbf{Adj}) + 3\sum_i (R_i - 1)T(\mathbf{r}_i) - K, \tag{2.33}$$

where $K\delta^{ab} = \frac{k_G}{2}\delta^{ab} - \mathrm{Tr}\,\gamma T^aT^b$. The second term involving γ is the contribution from the anomalous dimension of the non-Lagrangian sector. We now look at using this information to construct theories with Lagrangian and non-Lagrangian sectors.

2.4.2 Constructing Generalised Quiver Theories

In chapter 3 of this thesis we will be interested in constructing theories which have a non-Lagrangian sector and a Lagrangian sector and which have an RG flow. We will be interested in asking if there is an interacting IR fixed point and in determining the R-charge of the IR CFT and using it to ask (a) are there are any a-theorem violations and (b) are there are unitarity violating operators in the resulting theory. If the answer to either of these questions is yes then it is likely that we have not correctly determined the IR fixed point of the flow.

[3] Here we have used equation (2.13). This is valid since all the non-Lagrangian theories that we are considering are superconformal.

The a-theorem, originally proposed in [14] and recently proven in [15], relates to the Weyl anomaly coefficient, a, in equation (2.10). If we calculate this coefficient for the UV and IR fixed points then the a-theorem tells us that $a_{UV} > a_{IR}$ so that the coefficient in some sense counts degrees of freedom. Once we know the R-charge of the IR theory we can use it to calculate the anomaly coefficients using the relations [16, 17]

$$a = \frac{3}{32}\left[3\,\mathrm{Tr}\,R^3_{\mathcal{N}=1} - \mathrm{Tr}\,R_{\mathcal{N}=1}\right] \qquad c = \frac{1}{32}\left[9\,\mathrm{Tr}\,R^3_{\mathcal{N}=1} - 5\,\mathrm{Tr}\,R_{\mathcal{N}=1}\right]. \tag{2.34}$$

We can then use the first of these to check that $a_{UV} > a_{IR}$.

The unitarity violations that were referred to are simply related to the dimensions of gauge-invariant operators. A consequence of unitarity is that any gauge-invariant operators $\mathcal{O}$ satisfies $D[\mathcal{O}] \geq 1$ where the inequality is saturated for a free field. Equivalently since we will be analysing IR fixed points the unitarity constraint is $R[\mathcal{O}] \geq \frac{2}{3}$, where R is the superconformal R-charge.

Determining the IR R-symmetry is in general non-trivial. Even if we know the UV R-symmetry the IR R-symmetry could be different. This is because in the IR any anomaly-free $U(1)$ symmetries can mix with the R-charge so that the IR R-charge can be any linear combination of UV R-charge and anomaly-free $U(1)$ symmetries. In addition to this there are often so-called emergent symmetries that were not present in the UV theory but which emerge in the IR theory. These can also mix with the R-symmetry in the IR. There is however a unique superconformal R-symmetry which appears in the superconformal algebra and which is related to the dimensions of operators. This superconformal R-symmetry is determined by using a-maximisation which is presented in the next subsection.

As an example of the process of constructing generalised quiver theories we look briefly at the S_ℓ theories constructed in [18]. We start by taking an $\mathcal{N} = 2$ superconformal quiver theory which contains a chain of $SU(N)$ gauge groups connected by bifundamentals. For the theory to be conformal we could couple the first and last gauge group to N fundamental hypermultiplets. Instead we couple to the first and last gauge group a non-Lagrangian SCFT each of which contributes the same amount to the beta function for the gauge coupling of the first/last gauge group as N fundamental hypermultiplets. The generalised quiver diagram for this theory is given in figure 2.6 where the triangles denote non-Lagrangian SCFTs.

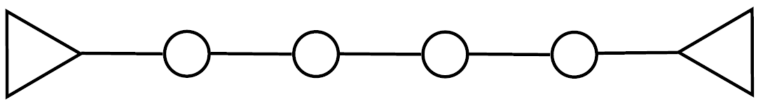

Figure 2.6: Theory with 4 $SU(N)$ gauge groups, 3 bifundamental hypermultiplets and an SCFT at each end of the quiver.

The interacting SCFTs that we have coupled to the first and last gauge group must have $SU(N)$ as a subgroup of the flavour symmetry where the central charge of this $SU(N)$ flavour symmetry which is $k_{SU(N)\subset G} = 2N$. This way the beta function of the first/last gauge group is (again using holomorphically normalised gauge fields)

$$\beta = 2T(\text{Adj})) - 2NT(\Box) - \frac{k_{SU(N)}}{2} = 2N - N - \frac{k_{SU(N)}}{2}, \tag{2.35}$$

which is equal to zero if $k_{SU(N)\subset G} = 2N$. The non-Lagrangian theories that were used in [18] were T_N theories, first found in [19]. These theories have an $SU(N)^3$ flavour symmetry and indeed $k_{SU(N)\subset SU(N)^3} = 2N$.

Once we have constructed these $\mathcal{N} = 2$ superconformal generalised quiver theories we will break to $\mathcal{N} = 1$. We will do this by deleting the adjoint $\mathcal{N} = 1$ chiral multiplets that live inside the $\mathcal{N} = 2$ vector multiplets. What we will be left with is a series of $\mathcal{N} = 1$ vector superfields coupled to hypermultiplets and non-Lagrangian SCFTs. Another way of doing this is to give a mass to the adjoint chiral superfield and taking the limit $m \to \infty$. We wish to analyse the resulting IR SCFT.

For $\mathcal{N} = 2$ SCFTs the R symmetry is an $SU(2)_R \times U(1)_R$ symmetry. For the fields that live in a hypermultiplet or vector multiplet the charges are tabulated below. In the table $R_{\mathcal{N}=2}$ is the $U(1)_R$ charge and I_3 is the third generator of the $SU(2)_R$ group.

$R_{\mathcal{N}=2}\backslash I_3$	$\frac{1}{2}$	0	$-\frac{1}{2}$
0		A_μ	
1	λ		λ'
2		ϕ	

$R_{\mathcal{N}=2}\backslash I_3$	$\frac{1}{2}$	0	$-\frac{1}{2}$
-1		ψ	
0	Q		$\widetilde{Q}^\dagger$
1		$\widetilde{\psi}^\dagger$	

If we are interested in the $\mathcal{N} = 1$ properties of these theories we will make use of the following symmetries:

$$\begin{aligned} R_{\mathcal{N}=1} &= \frac{1}{3}R_{\mathcal{N}=2} + \frac{4}{3}I_3, \\ J &= R_{\mathcal{N}=2} - 2I_3. \end{aligned} \tag{2.36}$$

This is one choice of linear combinations of the $U(1)$ symmetries that live inside the $SU(2)_R \times U(1)_R$ symmetry where J is a non-R symmetry and $R_{\mathcal{N}=1}$ is the UV exact $\mathcal{N} = 1$ superconformal R-symmetry.

When we start with the initial $\mathcal{N} = 2$ theory and give masses to the adjoint chiral superfields in the theory the R-symmetry that is preserved by this deformation is

$$R_0 = R_{\mathcal{N}=1} + \frac{1}{6}J = \frac{1}{2}R_{\mathcal{N}=2} + I_3. \tag{2.37}$$

This is one particular choice of IR R-symmetry for the S_ℓ theory however it is possible that anomaly-free $U(1)$ symmetries mix with this in the IR so that the superconformal R-symmetry is some linear combination. In addition to this R-symmetry there is one anomaly-free $U(1)$ symmetry:

$$\mathcal{F} = J_1 + (-1)^{\ell+1}J_2 + \sum_{i=1}^{\ell}(-1)^{i+1}F_i, \tag{2.38}$$

where J_1 is the $U(1)$ J current associated to the left-most T_N in the quiver, J_2 is the $U(1)$ J current associated to the right-most T_N and F_i is $U(1)$ J current associated to the i-th hypermultiplet (appropriately normalised so that the F-charge of the i-th hypermultiplet is 1).

For even ℓ this current has $\operatorname{Tr}\mathcal{F} = 0$ and so does not mix with the R symmetry (this is explained in the next subsection). This means that the IR R-symmetry is simply R_0.

When ℓ is odd $\operatorname{Tr}\mathcal{F} \neq 0$ and the symmetry does mix with the R symmetry so we have to use a maximisation to find the exact superconformal R symmetry. This was done in [18] and they found that the S_ℓ theories present no cases of unitarity violations or a-theorem violations.

2.4.3 a-maximisation

As described in the previous subsection the theories we plan to examine have some anomaly-free R symmetry which we have chosen but also have an additional anomaly-free $U(1)$ symmetry. We will always be able to choose the R symmetry to be some linear combination of an R symmetry and any additional anomaly-free $U(1)$ symmetries in the theory:

$$R_{trial} = R_0 + \sum_I s_I F_I, \tag{2.39}$$

where F_I are the additional anomaly-free $U(1)$ symmetries and s_I are parameters. In [20] it was shown that if we have some choice of $U(1)_R$ symmetries the unique superconformal R symmetry is the one that locally maximises the function

$$a_{trial} = 3\operatorname{Tr} R_{trial}^3 - \operatorname{Tr} R_{trial}. \tag{2.40}$$

The maximum value of this function is equal to the central charge a at the conformal point. We will use this tool in what follows.

If we have some anomaly-free $U(1)$ current, F_I, for which $\operatorname{Tr}(F_I) = 0$ then because of the extremum condition $9\operatorname{Tr} R^2 F_I = \operatorname{Tr} F_I$ we can always take the R-symmetry to commute with F_I[20].

2.5 Moduli Space

In this section we briefly describe the moduli space of vacua that preserve supersymmetry. For a more information on this subject see [9, 21, 22].

It is a generic feature of four-dimensional supersymmetric gauge theories that the potential is often independent of combinations of vev's. For this reason there will often be a space of vacua of the theory that preserve supersymmetry. This space of vacua is called the *moduli space*.

If $|0\rangle$ is a SUSY-invariant vacuum this means that

$$Q_\alpha|0\rangle = 0, \qquad \overline{Q}_{\dot{\alpha}}|0\rangle = 0. \tag{2.41}$$

From the SUSY algebra,

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu, \tag{2.42}$$

one can see that the Hamiltonian can be expressed as

$$H = P^0 = \frac{1}{4}\left(Q_1\overline{Q}_1 + \overline{Q}_1 Q_1 + Q_2\overline{Q}_2 + \overline{Q}_2 Q_2\right). \tag{2.43}$$

This means that any vacuum of a theory that preserves SUSY must have zero energy.

For a theory with superpotential, W, the scalar potential is given by the sum of the "D-terms" and the "F-terms":

$$V(\Phi_i) = \sum_a \frac{1}{2} D^a D^a + \sum_i |\partial_i W|^2, \tag{2.44}$$

where the second term is the derivative of the superpotential with respect to the i-th field and D^a is

$$D^a = \sum_i \Phi_i^* T^a_{\mathbf{r}_i} \Phi^i, \tag{2.45}$$

where $T^a_{\mathbf{r}_i}$ is the generator of the representation of the gauge group that Φ^i is in. We do not consider here theories with Fayet-Iliopoulos terms

The D-terms and F-terms are both positive definite and so in order to have a vacuum with zero energy we must have that the D- and F-terms both vanish. Solving these equations gives us the space of vacua that preserve supersymmetry. We must also, however, take into account vacua related by gauge symmetry. Only the space of gauge-inequivalent vacua is the true space of vacua of the gauge theory, the moduli space.

We will explicitly solve two relatively simple examples so that we can use the insights in later sections.

2.5.1 SQED

The SQED theory is a theory of N_F chiral superfields that are all charged under a $U(1)$ gauge group. The chiral superfields Φ_i have scalar components ϕ_i and their respective charges are denoted q_i. There is no superpotential for this theory and so the F-terms are automatically zero. Thus the moduli space is just the space of solutions to the D-term equations. The D-term equations are

$$q_i|\phi_i|^2 = 0. \tag{2.46}$$

In this equation it is understood that ϕ_i denotes the vev of the scalar components of the Φ_i chiral superfield. There are additional constraints coming from anomaly cancellation conditions which are $\sum_i q_i = \sum_i q_i^3 = 0$[4]. It is the space of solutions to equation (2.46) subject to anomaly constraints and modulo gauge transformations that is the moduli space of vacua of the theory.

For the simple case of 2 chiral superfields the anomaly constraint gives us that the two superfields have opposite charge $q_\pm = \pm 1$ so that a gauge transformation is given by:

$$\begin{aligned}\phi_+ &\to e^{i\alpha}\phi_+,\\ \phi_- &\to e^{-i\alpha}\phi_-.\end{aligned}$$

For this set of chiral superfields the vacuum equations are:

$$\begin{aligned}&|\phi_+|^2 - |\phi_-|^2 = 0\\ \Rightarrow &|\phi_+| = |\phi_-|\,.\end{aligned}$$

If we say $\phi_+ = re^{i\theta_1}$ then $\phi_- = re^{i\theta_2}$ and the moduli space is spanned by (r, θ_1, θ_2). However there are gauge transformations that relate different vacua in the moduli space. Under a gauge transformation a point in the moduli space transforms as:

$$(r, \theta_1, \theta_2) \to (r, \theta_1 + \alpha, \theta_2 - \alpha)\,.$$

Thus the gauge orbits are lines of constant $\theta_1 + \theta_2$. $(r, \theta_1 + \theta_2)$ then covers the entire moduli space modulo gauge transformations.

This means that the holomorphic gauge-invariant co-ordinate $\phi_+\phi_-$ spans the entire moduli space and can be used to identify different vacua.

2.5.2 SQCD

After doing the simple example of SQED we now more on to the slightly more complicated example of SQCD however we will only deal with this theory classically. The matter content of SQCD and the charges of the fields under the various symmetries are given in table 2.1. There is no superpotential and thus no F-terms. The D-term equations are

$$D^A = \left(T^A\right)_a^{\ b}\left(\overline{Q}_i^a Q_b^i - \widetilde{Q}_i^a \overline{\widetilde{Q}}_b^i\right) = 0. \tag{2.47}$$

For convenience we define the matrices $d_b^a = \overline{Q}_i^a Q_b^i$ and $\widetilde{d}_b^a = \widetilde{Q}_i^a \overline{\widetilde{Q}}_b^i$. The generators T^A along with the identity matrix form a basis of $N \times N$ matrices and as a consequence of this we get that

$$d - \widetilde{d} = \alpha \mathbb{1}. \tag{2.48}$$

[4]The second of these conditions is the familiar gauge anomaly that was presented in section 2.2 which must vanish for consistency of the theory. The first of these conditions is a anomaly constraint with one local current and two energy-momentum tensors. This is called the gauge-gravity anomaly and must also vanish for consistency of the theory.

There are then two different cases: $N_F < N_C$ and $N_F \geq N_C$.

$N_F < N_C$

For $N_F < N_C$ the rank of d is at most N_F and the same goes for $\widetilde{d}$. We can use an $SU(N_C)$ gauge transformation to put d and $\widetilde{d}$ in diagonal form so that they both have at most N_F non-zero eigenvalues:

$$d = \begin{pmatrix} v_1^2 & & & & & & \\ & v_2^2 & & & & & \\ & & \ddots & & & & \\ & & & v_{N_F}^2 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}, \quad \widetilde{d} = \begin{pmatrix} \widetilde{v}_1^2 & & & & & & \\ & \widetilde{v}_2^2 & & & & & \\ & & \ddots & & & & \\ & & & \widetilde{v}_{N_F}^2 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}. \tag{2.49}$$

From this we see that $\alpha = 0$ and so $d = \widetilde{d}$. Then we can use the $SU(N_F)^2$ flavour symmetry to put Q and $\widetilde{Q}$ in diagonal form and since $d = \widetilde{d}$ we have

$$Q = \overline{\widetilde{Q}} = \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_{N_F} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}. \tag{2.50}$$

Thus for a generic vacuum the $SU(N_C)$ symmetry breaks to $SU(N_C - N_F)$ meaning that $(N_C^2 - 1) - ((N_C - N_F)^2 - 1) = 2N_F N_C - N_F^2$ gauge bosons acquire mass. We originally had $2N_F N_C$ chiral superfields however $2N_F N_C - N_F^2$ of these will be eaten by the massive vector superfields. This means that there will be N_F^2 chiral superfields left which can be given a vev and thus there are N_F^2 massless directions.

Analogously to the SQED case we can define holomorphic gauge-invariant co-ordinates that span the moduli space:

$$M_i^j = \widetilde{Q}_i^a Q_a^j = \widetilde{Q}_i^a \overline{\widetilde{Q}}_a^j = \overline{Q}_i^a Q_a^j.$$

This confirms that the dimension of the moduli space is indeed N_F^2.

$N_F \geq N_C$

For the case where $N_F > N_C$ we again get that $d - \widetilde{d} = \alpha \mathbb{1}$. Again we can do an $SU(N_C)$ transformation so that d is diagonal. This time d is of maximal rank N_C so generally it has no zero eigenvalues:

$$d = \begin{pmatrix} |v_1|^2 & & \\ & \ddots & \\ & & |v_{N_C}|^2 \end{pmatrix},$$

and again because we have that $d - \widetilde{d} = \alpha \mathbb{1}$ we can say that $\widetilde{d}$ is diagonal as well with eigenvalues $|\widetilde{v}_i|^2$. The vacuum equations then become:

$$|v_i|^2 = |\widetilde{v}_i|^2 + \alpha.$$

Again since d and $\widetilde{d}$ are invariant under flavour transformations we can put Q^i_a and $\overline{\widetilde{Q}}^i_a$ in diagonal form:

$$Q^i_a = \begin{pmatrix} v_1 & & & 0 & \\ & \ddots & & & \ddots & \\ & & v_{N_C} & & & 0 \end{pmatrix}.$$

$$\widetilde{Q}^a_i = \begin{pmatrix} \overline{v}^*_1 & & \\ & \ddots & \\ & & \overline{v}^*_{N_C} \\ 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}.$$

In this case the $SU(N_C)$ gauge symmetry has been completely broken so $N_C^2 - 1$ gauge superfields have acquired mass leaving $2N_C N_F - N_C^2 + 1$ massless directions. In this case the holomorphic gauge-invariant co-ordinates are:

$$\begin{aligned} M^i_j &= Q^i \widetilde{Q}_j \\ B^{i_1 \dots i_{N_C}} &= Q^{i_1}_{a_1} \dots Q^{i_{N_C}}_{a_{N_C}} \epsilon^{a_1 \dots a_{N_C}} \\ \widetilde{B}_{i_1 \dots i_{N_C}} &= \widetilde{Q}^{a_1}_{i_1} \dots \widetilde{Q}^{a_{N_C}}_{i_{N_C}} \epsilon_{a_1 \dots a_{N_C}}. \end{aligned}$$

It should be noted however that there seem to be $2\binom{N_F}{N_C} + N_F^2$ holomorphic gauge-invariant co-ordinates for the $2N_C N_F - N_C^2 + 1$ dimensional moduli space. The reason for this is that these co-ordinates form an over-complete basis and they are subject to various constraints.

2.6 Conformal Manifold

Another important concept that arises in the study of quantum field theories is the conformal manifold. For any quantum field theory with a marginal operator, $\mathcal{O}$ we may add the term $h\mathcal{O}$ to the Lagrangian to get a new theory which is also conformal. This defines a conformal line and for multiple marginal operators there will be a conformal manifold. Marginality of the operator $\mathcal{O}$ at $h = 0$ does not necessarily imply that $\mathcal{O}$ is marginal at $h \neq 0$ since the dimension of $\mathcal{O}$ may depend on h. For $\mathcal{N} = 1$ theories [23] (or equivalently [24]) gives a method for computing the dimension of the conformal manifold which we now briefly describe.

If we have some operator, $W = h\phi_1 \ldots \phi_n$, which can be added to the superpotential then the beta function for this term is

$$\beta_h = \frac{\partial h(\mu)}{\partial \ln \mu} = h(\mu)\left(-d_W + \sum_i \left[d(\phi_i) + \frac{1}{2}\gamma(\phi_i)\right]\right) = h(\mu)A_h, \tag{2.51}$$

where d_W is the canonical dimension of the superpotential, $d(\phi_i)$ is the canonical dimension of ϕ_i and $\gamma(\phi_i)$ is the anomalous dimension of ϕ_i.

Similarly, for a gauge coupling constant the beta function is given by

$$\beta_g = \frac{\partial g(\mu)}{\partial \ln \mu} = -f(g[\mu])\left(\left[3T(\mathbf{Adj}) - \sum_i T(\mathbf{r}_i)\left(1 - \gamma_i(g)\right)\right]\right) = f(g[\mu])A_g. \tag{2.52}$$

Here we have defined the scaling coefficients A_g and A_h which must equal zero for the theory to be conformal.

If we have a theory with n gauge coupling constants and m marginal operators then we automatically have $n + m$ beta functions that need to be set to zero for the theory to be conformal. These beta functions however need not be linearly dependent and so if we only have $k < n + m$ linearly independent beta functions then if any solution exists to the equations setting the beta functions to zero the solution space will be $(n + m - k)$-dimensional.

As an example of a theory with a conformal manifold we consider a typical two dimensional Landau-Ginsburg model with n identical chiral superfields, ϕ_i and superpotential $W = \lambda \sum_i (\phi_i)^n$. Since $d_W = 1$ and $d_i = 0$ the beta function for the superpotential satisfies $\beta_\lambda \propto -1 + \frac{1}{2}n\gamma(\lambda)$. λ is the anomalous dimension of each of the chiral superfields which are all equal by symmetry. Thus if a conformal fixed point exists then we have $\gamma(\lambda^*) = \frac{2}{n}$ at this point.

We now consider adding the marginal operator $\delta W = h\phi_1 \ldots \phi_n$ to the superpotential. The beta function for this superpotential term satisfies $\beta_h \propto -1 + \frac{1}{2}\sum_i \gamma(\lambda, h) \propto \beta_\lambda$ and so the two beta functions are linearly dependent. This means that if there is a solution to the equations setting the beta functions to zero the solution space is one dimensional, i.e. the space (λ, h) subject to $\gamma(\lambda, h) = \frac{2}{n}$.

The prescription outlined in [23] for determining the dimension of the conformal manifold is: add the number of gauge coupling constants to the number of marginal operators and subtract from this the number of independent anomalous dimensions. We will use this method later in the thesis to determine the dimension of the conformal manifolds of some of the theories that we study.

As was mentioned at the beginning this chapter it is intended as an introduction to some basic concepts which are taken as understood later in the thesis. Introductions to more advanced topics will be given as needed in later chapters.

Chapter 3

Theories of Class $\mathcal{S}$ and New $\mathcal{N} = 1$ SCFTs

This chapter is based on the paper [1].

3.1 Introduction

We are living in a golden age of quantum field theories. The diversity of theories available to study is astonishing, and due to the technological advances of recent years, many strongly coupled theories that had been considered intractable are now able to be investigated. There are no better examples of this than the supersymmetric compactifications of the still-mysterious six-dimensional $(2,0)$ theory. These exotic theories, which generically do not have free-field limits, are nevertheless rather understandable, and many quantities of interest (e.g. operator dimensions) are calculable. Although a great deal of progress has been made on compactifications of the $(2,0)$ theory to two and three dimensions, in the present work we will be most interested in the four-dimensional theories that come from compactifying the $(2,0)$ type A theory on a (punctured) Riemann surface. This compactification can be done in such a way as to preserve $\mathcal{N} = 2$ SUSY in four dimensions [25], and the resulting theories are called theories of class $\mathcal{S}$.

For theories of class $\mathcal{S}$ and type A, we need only specify two pieces of compactification data in order to determine the theory: the genus g of the Riemann surface, and the pole structures of the punctures. Since any (punctured) Riemann surface with genus $g > 1$ can be described with a suitable gluing of thrice-punctured spheres, we can describe any such theory by a set of these spheres, (called "fixtures" in [26]) with some subset of punctures connected by cylinders, so that the final object has the required genus and punctures. The most famous examples of fixtures, and the flagship examples for novel four-dimensional SCFTs, are Gaiotto's T_N theories [19]. Using this construction, the punctures correspond to global symmetries, and the cylinders correspond to gauge symmetries. In this manner, we can construct infinitely many $\mathcal{N} = 2$ SCFTs.

Because of the tremendous amount of freedom available to us in constructing such theories, the landscape of theories of class $\mathcal{S}$ still seems like the Wild West, and although general principles for these theories are known, not overly many specific examples have been explored. Much as in the case with D3-branes at the tip of a toric singularity, it would be useful to have a nice infinite family of theories to play with, like the $Y_{p,q}$ or $L_{p,q,r}$ theories. In this chapter, we point out the existence of such an infinite family, which includes and generalises Gaiotto's T_N theories. For reasons that will become apparent in the body of the chapter, we refer to these as the $T_{N,k}$ theories.

Many properties of these theories are still mysterious. For theories of class $\mathcal{S}$, although much is known about the Coulomb branch via the Seiberg-Witten curve, the Higgs branch remains relatively unexplored. One reason is because, unlike in theories with Lagrangian descriptions, there is no candidate basis of UV-free fields one could use to build a list of Higgs branch operators. Thus, it remains unclear how to even find the Higgs branch operators, much less the intricate relationships between them.

One window we do have into the Higgs branch is through the superconformal index (SCI) [27, 28], which is a useful tool for finding operators. In theories of class $\mathcal{S}$, a reduced version of the SCI was found in [29], and it is possible to use this to infer the existence of some Higgs branch operators which are difficult to see from duality alone, along with some of their quantum numbers.

Although it is far from obvious from the fixtures-and-punctures approach, we can similarly construct a huge variety of new $\mathcal{N} = 1$ theories. Geometrically, one way of doing this is to change the embedding of the Riemann surface in the 11-dimensional space by suitably twisting the normal bundle. The existence of certain $\mathcal{N} = 1$ supergravity solutions was first shown in [25]; these solutions were then shown to be part of a much larger set of solutions in [30], and further supergravity solutions were found in [31]. Alternately, one could use a recently-discovered class of punctures [32, 33, 34, 35] which preserve only $\mathcal{N} = 1$ SUSY. From a field theory perspective, although certain of these solutions arise at the endpoints of flows from theories of class $\mathcal{S}$ [4], the overwhelming majority are not known to do so.

Another goal of the present work is to further the study of $\mathcal{N} = 1$ theories built out of class $\mathcal{S}$ fixtures, as begun in [4] and continued in [18, 30, 32, 33, 34, 36, 37, 38, 39, 40, 41]. The study of these $\mathcal{N} = 1$ theories is still in its infancy, and many of their properties are unknown. In particular, it is not in general known which such theories are superconformal, and just as in conventional gauge theories, finding the IR phase of a given theory is often a difficult process. In [18], several such theories were analysed, and flows between them were used to establish evidence for the existence or non-existence of the conformal fixed points. In the present work we re-examine these flows, and find a subtlety in the previous analysis which indicates that some of the theories not previously believed to flow to interacting conformal points may in fact do so.

The remainder of the chapter is structured as follows. In Section 3.2, we review some basic class $\mathcal{S}$ technology. In Section 3.3, we introduce a particularly interesting subfamily of theories of class $\mathcal{S}$, the $T_{N,k}$'s, and describe some of their properties. In Section 3.4, we review the superconformal index, and use it to elucidate further properties of the $T_{N,k}$ theories. In Section 3.5, we construct $\mathcal{N} = 1$ theories from the $T_{N,k}$'s, and describe some flows between them. Finally, in Section 3.6, we describe some initial attempts to construct theories in the manner of [36]. Various results are collected in appendices.

3.2 Review

Even though many of the results in this work will be for $\mathcal{N} = 1$ theories, we will need to begin by reviewing some relevant $\mathcal{N} = 2$ technology. This will allow us to construct an interesting subclass of theories, which we will then explore in the remainder of the chapter.

3.2.1 Theories of Class $\mathcal{S}$

We begin with a brief review of theories of class $\mathcal{S}$. This subsection roughly follows the format of [26]. These theories are obtained by compactifying the six-dimensional $(2,0)$ theory on a Riemann surface $\mathcal{C}$ with punctures. In this work we only consider theories coming from type A_{N-1} six-dimensional $(2,0)$ theories; these theories arise on the world-volume of a stack of N M5-branes.

In [19], Gaiotto showed that the space of marginal couplings of these theories could be identified with the moduli space of a curve $\mathcal{C}_{g,h}$ with genus g and h punctures. Since then, these theories have seen a great deal of study, and it has been observed that the parameters defining the four-dimensional theory are completely determined by the two-dimensional compactification surface. These defining parameters of the theory are, in addition to the genus g of $\mathcal{C}_{g,h}$, the location and type of the punctures on the surface. This data is encoded in the Seiberg-Witten curve, which is of the form $\lambda^N = \sum_{k=2}^{N} \lambda^{N-k}\phi_k$, where λ is the Seiberg-Witten differential and ϕ_k are k-differentials $(k = 2, ..., N)$. The ϕ_k will, in general, have poles at each of the punctures. Each puncture then can be characterised by its pole structure $\{p_k\} = \{p_2, p_3, ..., p_n\}$ where p_k is the order of the pole that ϕ_k has at the puncture. Then, for a given surface $\mathcal{C}_{g,h}$ we can specify both the number of punctures as well as their individual pole structures.

Punctures come in two varieties, regular and irregular; which category a given puncture is in is determined by its pole structure. A *regular* puncture is a puncture to which we can assign a Young tableau using the following rules[1]

- Draw a Young tableau with two boxes in a row;
- For each $k = 3, ..., N$, if $p_k = p_{k-1} + 1$ add a box to the current row, and if $p_k = p_{k-1}$ start a new row with one box.

[1]More generally for theories of class $\mathcal{S}$, regular punctures are classified by embeddings of $SU(2)$ in the ADE Lie algebra of the six-dimensional theory. For class $\mathcal{S}$ theories of type A we can use Young tableaux, however, for type A theories in the presence of an outer automorphism twist, or type D or E theories, this will not suffice. For more information see [42, 43, 44, 45, 46, 47].

All regular punctures must have $p_2 = 1$. As an example of a regular puncture, we have drawn the Young tableau in figure 3.1, for the puncture with pole structure $\{p_k\} = \{1,2,3,4,5,6,7,7,8,9,10,11,12,12,13,14,15,16,16,17,18,19,20,20,21\}$. The associated flavour symmetry is $SU(3)\times SU(2)^2 \times U(1)^3$.

The non-R global (flavour) symmetry group associated to a puncture is given by $G = S\left(\prod_h U(n_h)\right)$, where the product is over column heights of the Young tableau and n_h is the number of columns with height h. The $S(...)$ means an overall $U(1)$ is removed. There are two special regular punctures worth highlighting. The first is a maximal puncture which has pole structure $\{1,2,...,N-1\}$ and flavour symmetry $SU(N)$; the corresponding Young tableau has one row of N boxes. The second is a minimal puncture, which has pole structure $\{1,1,...,1\}$ and flavour symmetry $U(1)$; the corresponding Young tableau has one row of 2 boxes and $N-1$ rows of 1 box each.

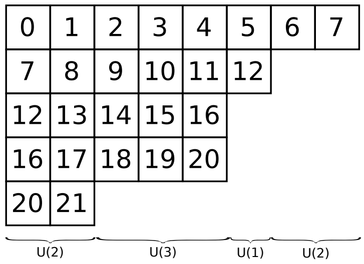

Figure 3.1: A Young tableau for a regular puncture; the flavour symmetry associated to it is $S\left(U(3)\times U(2)^2\times U(1)\right)$.

Irregular[2] punctures are those punctures which do not satisfy the conditions for regular punctures, but do satisfy a different set of conditions whose structure we do not detail here; for useful discussions on irregular punctures see [43] or [49].

Fixtures and Cylinders

A *fixture* is a thrice-punctured sphere specified by the pole structure of each of the punctures. The quantity

$$d_k = 1 - 2k + \left(\sum_{i=1}^{3} p_k^{(i)}\right), \tag{3.1}$$

where the sum is over the punctures, gives us the number of Coulomb branch operators of dimension k. We can thus find the dimension of the Coulomb branch by summing over $k = 2,...,N$. If the dimension of the Coulomb branch is zero then the fixture corresponds to a set of free hypermultiplets, and if the dimension of the Coulomb branch is greater than zero, then the fixture corresponds to a "non-Lagrangian" SCFT[3], or a combination of a non-Lagrangian SCFT and free hypers. Although the flavour symmetry of a fixture is usually just the product of the flavour symmetries associated to each of the punctures, there are some cases where the symmetry enhances.

One example of a fixture is one with two maximal punctures and one minimal puncture. This fixture has $d_k = 0$ for all $k = 2,...,N$ and corresponds to a theory of N^2 free hypermultiplets. A second useful example is the fixture with three maximal punctures.

[2] "Irregular" here is used in the sense of [26], and not in the same sense as most of the Hitchin system literature, e.g. [48].

[3] As usual, the phrase "non-Lagrangian" merely means that no free-field UV description is known to exist, and not that such a description has been conclusively ruled out.

This fixture corresponds to the T_N theory [19]. The T_N has flavour symmetry $SU(N)^3$, when $N > 3$. The case $N = 3$ is the E_6 SCFT of [50], and $N = 2$ is a theory of 4 free hypermultiplets. The graded dimension[4] of the Coulomb branch for the T_N is $\{d_k\} = \{0, 1, 2, 3, ..., N - 2\}$.

Punctures can be connected via *cylinders*, which correspond to a gauge group G which must be a subgroup of the flavour symmetry group associated to each of the two punctures that it is connecting; this corresponds to gauging a flavour symmetry. As the cylinders get longer, the corresponding gauge coupling becomes weaker. Even for class $\mathcal{S}$ theories of the same type, not every pair of punctures admits a cylinder connecting them; for the complete rules for type A theories, see [26].

S-Duality

From the perspective of punctured surfaces, S-duality corresponds to different degeneration limits into thrice-punctured spheres connected by cylinders. As an example we look at the case of Argyres-Seiberg duality [12], which is depicted in Figure 3.2. The theory in

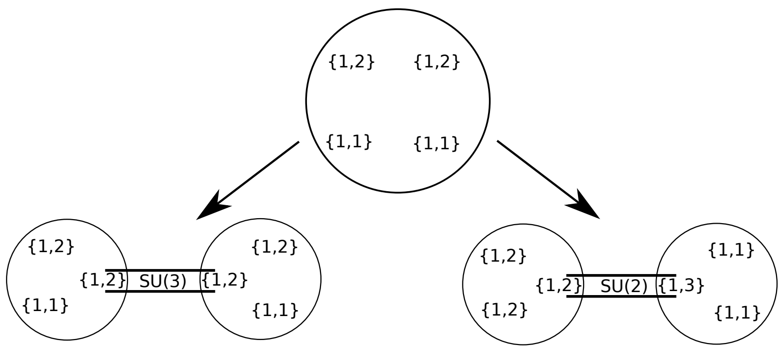

Figure 3.2: The two degeneration limits of a punctured sphere with two maximal punctures and two minimal punctures. The picture on the left corresponds to the degeneration limit corresponding to an $SU(3)$ gauge theory with 6 fundamental hypermultiplets. The picture on the right corresponds to an $SU(2)$ gauge theory with one fundamental hypermultiplet and the E_6 SCFT, where an $SU(2)$ subgroup of the E_6 flavour symmetry is gauged.

question is derived by wrapping the six-dimensional $(2, 0)$ A_2 theory on a Riemann surface with two maximal and two minimal punctures.

This theory can be decomposed into thrice-punctured spheres connected by cylinders in two ways. In one limit, there is an $SU(3)$ gauge theory with six hypermultiplets. In the other limit, there is an $SU(2)$ gauge theory with one hypermultiplet, where the $SU(2)$ gauges part of the global symmetry of the E_6 SCFT of [50]. Dualities of this form obey a set of consistency checks that were set out in [51].

Gaiotto duality is another example of S-duality and relates an $SU(N)^{N-2}$ gauge theory to a non-Lagrangian theory, the T_N, coupled to a Lagrangian "superconformal tail" (more about this in the next section). This duality can be seen as two ways in which a genus 0

[4] Graded dimension here means graded by operator dimension.

curve with two maximal punctures and $N-1$ minimal punctures can degenerate. These two ways are shown in figure 3.3.

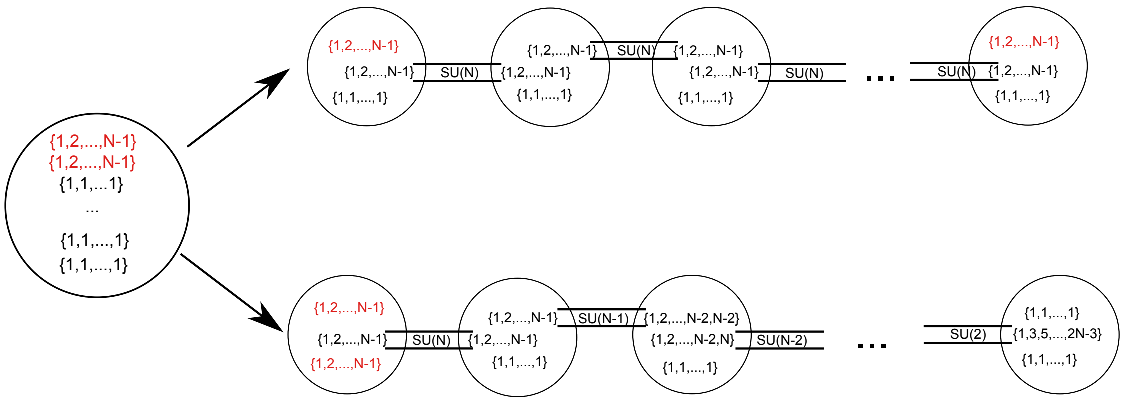

Figure 3.3: Two different degeneration limits of a Riemann surface with two maximal punctures and $N-1$ minimal punctures. On the top is an $SU(N)^{N-2}$ gauge theory with bifundamental hypermultiplets. Each fixture by itself corresponds to N^2 free hypermultiplets and each cylinder corresponds to an $SU(N)$ gauge group which weakly gauges the flavour symmetries of the hypermultiplets. On the bottom is the T_N coupled to a superconformal tail.

3.3 The $T_{N,k}$ Theories

In [19], evidence was given for the existence of a one-parameter family of $\mathcal{N} = 2$ SCFTs, the T_N theories. These theories are the low-energy energy limit of a stack of N M5 branes wrapping a sphere with three maximal punctures. In the present work, we consider a related class of theories which will display a variety of interesting properties. These theories, which we will call $T_{N,k}$, come from N M5-branes wrapping a sphere with two maximal punctures and a third puncture with pole structure $\{1, 2, 3, ..., k-1, k, k, ..., k\}$. The flavour symmetry of this theory is then $SU(N)^2 \times SU(k) \times U(1)$.

These theories form part of an interesting S-duality which is shown in figure 3.4. This duality corresponds to different ways in which a curve with two maximal punctures and k minimal punctures can degenerate into thrice-punctured spheres connected by cylinders. This set of S-dualities generalises the Gaiotto duality found in [19]. Gaiotto duality (see the middle row of figure 3.4) relates an $SU(N)^{N-2}$ gauge theory with bifundamental hypermultiplets to a T_N coupled to a superconformal tail (i.e. an $SU(N)\times SU(N-1)\times\cdots\times SU(2)$ gauge theory with bifundamental hypermultiplets). A related natural question to ask is what theory is S-dual to an $SU(N)^{k-1}$ linear quiver gauge theory for general k. For the case $k > N-1$ one can see that the dual theory is again a T_N coupled to a Lagrangian theory. This time the Lagrangian part is an $SU(N)^{k-N+1} \times SU(N-1) \times \cdots \times SU(2)$ gauge theory, as in the bottom row of figure 3.4. However, for the case $k < N-1$, we find that the dual theory is a $T_{N,k}$ coupled to a superconformal tail, as in the top row of figure

3.4.

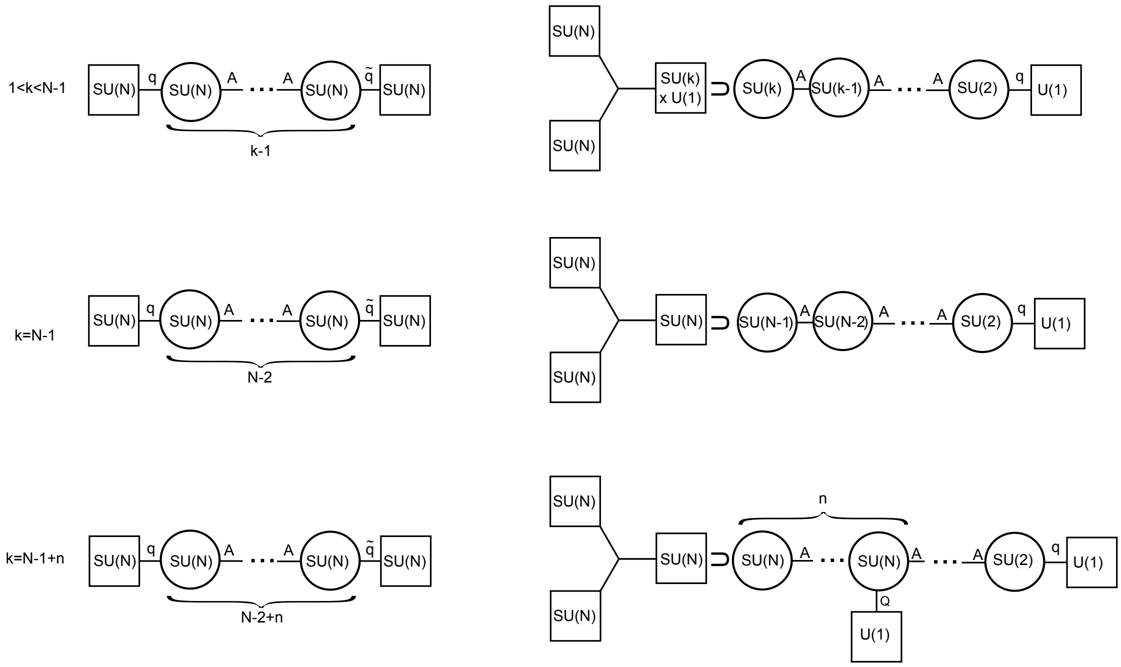

Figure 3.4: Duality between $\mathcal{N} = 2$ linear quiver gauge theories (left) and $T_{N,k}$ theories coupled to an $\mathcal{N} = 2$ superconformal tail (right). Circles represent gauge symmetries, boxes represent flavour symmetries, and lines represent bifundamental hypers. Trivalent vertices represent $T_{N,k}$ theories. $\supset$ represents gauging of a subgroup of a flavour symmetry. In one duality frame, we have an $SU(N)^{k-1}$ gauge theory with bifundamental hypermultiplets. In the other frame, we have a $T_{N,k}$ coupled to a quiver theory with gauge groups of decreasing rank. In the case of $k = 1$, we have N^2 free hypermultiplets in both duality frames, and for the case of $k = N - 1$ we have a T_N coupled to a superconformal tail. For all $k > N-1$ we have a T_N coupled to $SU(N)^{k-N+1} \times SU(N-1) \times SU(N-2) \times \cdots \times SU(2)$ gauge theory.

These theories also appear in another duality. In section 3.2.1, we reviewed how Gaiotto duality corresponds to two ways in which a genus 0 curve with two maximal punctures and $N - 1$ minimal punctures can degenerate into thrice-punctured spheres connected by cylinders (see figure 3.3). The first way is to have one maximal puncture on each end sphere; this corresponds to the $SU(N)^{N-2}$ gauge theory. The second is to have the maximal punctures both on one end, which then corresponds to the T_N coupled to the conformal tail. However, we can ask what happens when we degenerate the curve in such a way that the maximal punctures appear on fixtures in the middle of the curve rather than at the ends. In this case, the corresponding SCFT is a generalised quiver theory which involves $T_{N,k}$ theories, and the corresponding quiver diagram is as shown in figure 3.5.

One can also ask what happens for the different degeneration limits of a Riemann surface with 2 maximal punctures and k minimal punctures (*i.e.*, the other S-dual frames of the theories in figure 3.4). These different limits can be described by $T_{N,k}$'s and $\mathcal{N} = 2$

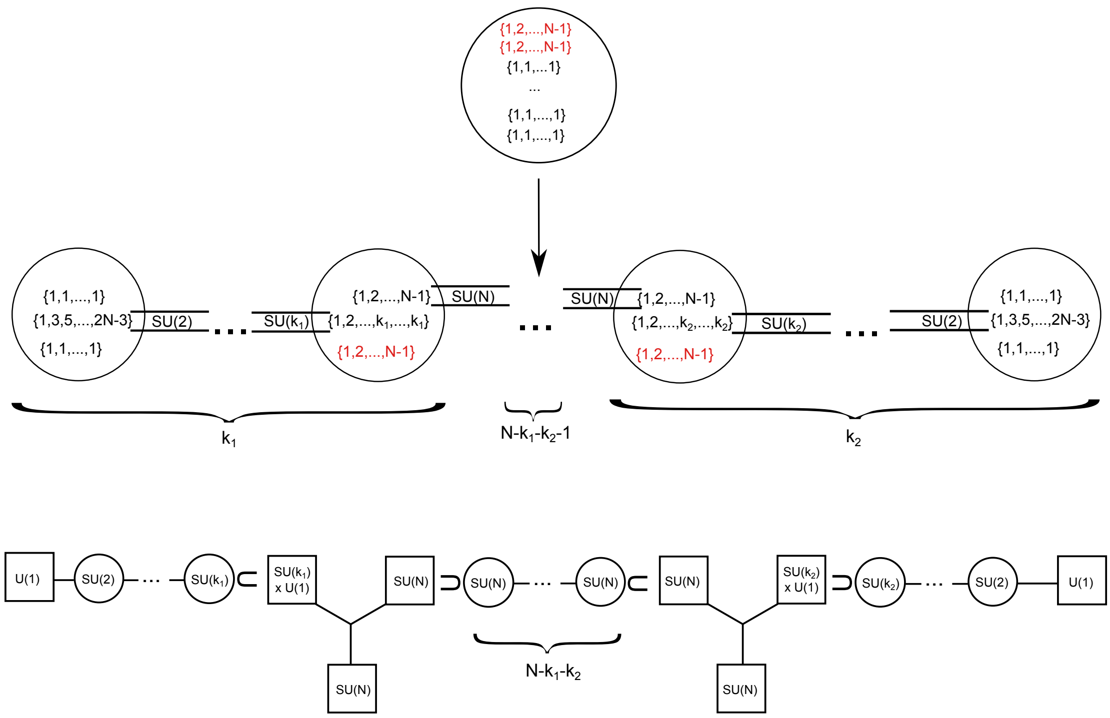

Figure 3.5: Top: the degeneration limit of a surface with two maximal punctures and $N-1$ minimal punctures into thrice-punctured spheres connected by cylinders. The maximal punctures appear on the k_1-th sphere from the left and the k_2-th sphere from the right. Bottom: the quiver diagram for the corresponding theory, which contains a T_{N,k_1} and a T_{N,k_2}.

vector and hyper multiplets.

There are a few special cases of the $T_{N,k}$'s worth mentioning, which we now go through in order of increasing k. First, as can be seen from the pole structure, the $T_{N,1}$ theory corresponds to N^2 free hypermultiplets. For $k=2$, the $T_{N,2}$ theory has its flavour symmetry enhanced to $SU(2N)\times SU(2)$. For $k=N-1$, $T_{N,N-1}$ is identically the T_N, so the flavour symmetry is enhanced to $SU(N)^3$. As can be seen from the pole structure of the third puncture, we cannot have $k>N-1$. We further note that the $T_{N,2}$, $T_{N,3}$ and $T_{N,4}$ theories feature in [26], where they are called $R_{0,N}$, U_N, and W_N respectively.

We can easily obtain the graded dimensions of the Coulomb branch using equation (3.1); these turn out to be $(d_2, d_3, ..., d_N) = (0,1,2,3,...,k-2,k-1,k-1,..., \ \ k-1)$. We also could have calculated this by looking at the duality in figure 3.4 and noting that d_k is the same in either duality frame. Since we know d_k for the linear quiver theory, we can just subtract the number of Coulomb branch operators of the superconformal tail from the whole dual theory to get d_k for the $T_{N,k}$.

The central charges for these theories are

$$a_{T_{N,k}} = \frac{(6k-5)\,N^2 - 2k^3 - \frac{5}{2}k^2 - \frac{1}{2}k + 5}{24}, \quad c_{T_{N,k}} = \frac{(3k-2)\,N^2 - k^3 - k^2 + 2}{12}, \quad (3.2)$$

again obtained by subtracting the central charges of the superconformal tail from those of the whole dual theory. For $k = N - 1$, these recover the known expressions for T_N. By similar methods, we can compute the leading coefficient of the two-point function of the flavour currents, also known as the central charge k_G for a flavour symmetry G. When we gauge a flavour symmetry, as we will do later, this quantity appears in the beta function of the associated coupling (for more info see [4]). For either $SU(N)$ the central charge is $k_{SU(N)} = 2N$, and for $SU(k)$, the central charge is $k_{SU(k)} = 2(k + 1)$.

3.3.1 Higgs Branch Operators

Our knowledge of the Higgs branch of theories of class $\mathcal{S}$ is still quite incomplete. Although some Higgs branch operators are known, and in some special cases we can make concrete statements, our knowledge of such operators is limited. In this subsection we review some relevant facts about Higgs branch operators for the T_N theories, and then use similar arguments to establish the existence of analogous operators for the $T_{N,k}$ theories. We leave the much more difficult question of the structure of the Higgs branch to future work; our goal here is merely to describe some of its operators, and not the various relations between them.

The authors of [52] give an argument for the existence of certain Higgs branch operators, which goes as follows. Consider the $SU(N)^{N-1}$ linear quiver with bifundamental hypermultiplets (*i.e.*, the bottom left quiver of figure 3.4 with $n = 1$). There is a gauge-invariant operator $H_{ij} = q_i A_1 A_2 ... A_{N-2} \widetilde{q}_j$ where q_i and $\widetilde{q}_j$ are the (fundamental) quarks, and the A's are the bifundamentals; i and j are flavour indices. This operator transforms in the $(\mathbf{N}, \mathbf{N})$ representation of the $SU(N)^2$ flavour symmetry and has dimension N. In the dual frame where the T_N is coupled to a superconformal tail, this operator can be written as $H_{ij} = \mathcal{O}_{ijk} Q^k$, where Q^k is the quark that transforms in the fundamental representation of the $SU(N)$ gauge group and $\mathcal{O}_{ijk}$ is an dimension-$(N - 1)$ operator in the $(\mathbf{N}, \mathbf{N}, \mathbf{N})$ representation of the $SU(N)^3$ flavour symmetry of the T_N. This trifundamental is one of the Higgs branch operators in the T_N theory, and a similar tri-antifundamental operator exists as well.

It is also worth considering what happens in linear quivers with different numbers of nodes. First, consider the $SU(N)^{N-2}$ linear quiver. Here, the gauge-invariant operator of interest is $H_{ij} = q_i A_1 A_2 ... A_{N-3} \widetilde{q}_j$, which has dimension $N - 1$. The dual frame corresponds to a T_N where one of the $SU(N) \subset SU(N)^3$ flavour symmetries has an $SU(N - 1)$ subgroup gauged. In this case, also discussed in [52], the operator H_{ij} can be identified with $\mathcal{O}_{ijN}$, which is the part of $\mathcal{O}_{ijk}$ which transforms as a singlet under the $SU(N - 1)$ gauge group. For the dualities denoted by $k = N - 1 + n$ in figure 3.4, the analogous operator can be written in the dual frame as $H_{ij} = \mathcal{O}_{ijk}(A_1 A_2 ... A_{n-1} Q)^k$, where we have hidden most gauge group indices.

As is well known, the existence of the $\mathcal{O}_{ijk}$ operators in the T_N theory explains the enhancement of the $SU(N)^3$ flavour symmetry to E_6 for the case of $N = 3$. In this case, the $\mathcal{O}_{ijk}$ operator is dimension two and contains a conserved current in its multiplet. Because the adjoint representation of E_6 decomposes under $E_6 \to SU(3)^3$ as

$$\mathbf{78} \to (\mathbf{8},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{8}) \oplus (\mathbf{3},\mathbf{3},\mathbf{3}) \oplus \left(\overline{\mathbf{3}},\overline{\mathbf{3}},\overline{\mathbf{3}}\right), \tag{3.3}$$

we see that the operators $\mathcal{O}^{ijk}$ (and $\overline{\mathcal{O}}_{\bar{i}\bar{j}\bar{k}}$) combine with the currents of the $SU(3)^3$ flavour symmetry to lift the symmetry to E_6.

We can similarly argue for the existence of certain Higgs branch operators in the $T_{N,k}$ theories. If we look at the $SU(N)^{k-1}$ linear quiver, there is a dimension k operator $H_{ij} = q_i A_1 ... A_{k-2} \widetilde{q}_j$ that transforms in the $(\mathbf{N},\mathbf{N})$ representation of the $SU(N)^2$ flavour symmetry. As described above, for the T_N, there are two arguments for the existence of the operators $\mathcal{O}_{ijk}$, one relying on the existence of a quark in the dual theory (when $k=N$) and one relying on a subgroup of the $SU(N)$ flavour symmetry being gauged (when $k = N-1$). However, for $k < N-1$ there is no quark in the dual theory, nor is there a gauged subgroup of the $SU(k)$. Instead we argue that the dual operator is an operator $H_{ij} = \mathcal{O}_{ij}$ that transforms in the $(\mathbf{N},\mathbf{N},\mathbf{1})$ of the $SU(N)^2 \times SU(k)$ flavour symmetry of the $T_{N,k}$ and has dimension k.

One can easily see that this is the case for $k=1$, where the $T_{N,1}$ corresponds to free hypermultiplets. In this case the operator $\mathcal{O}_{ij}$ is dimension one, and corresponds to the free hypermultiplets themselves. When $k=2$, these operators are dimension two. Since this case has an enhanced flavour symmetry, from $SU(N)^2 \times SU(2) \times U(1)$ to $SU(2N) \times SU(2)$, we expect that the $\mathcal{O}_{ij}$ has in its multiplet the conserved currents necessary to exhibit this enhancement. The adjoint representation of $SU(2N)$ decomposes under $SU(2N) \to SU(N)^2 \times U(1)$ as

$$\mathbf{4N^2-1} \to \left(\mathbf{N^2-1},\mathbf{1}\right)_0 \oplus \left(\mathbf{1},\mathbf{N^2-1}\right)_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus \left(\mathbf{N},\overline{\mathbf{N}}\right)_2 \oplus \left(\overline{\mathbf{N}},\mathbf{N}\right)_{-2}, \tag{3.4}$$

so the $\mathcal{O}_{ij}$ and $\overline{\mathcal{O}}^{ij}$ are exactly what we need to enhance the flavour symmetry. In the next section, we will further bolster the case for the existence of these operators by showing that the $\mathcal{O}_{ij}$ appear in the superconformal index.

Along with the $\mathcal{O}_{ijk}$, the T_N also contains three dimension-two Higgs branch operators μ_i, $i=1,2,3$, that transform in the adjoint representation of each $SU(N)$. These operators are necessarily coupled to any relevant vector multiplets for gauged flavour symmetries via a superpotential $W = \mu\Phi$, as required by $\mathcal{N}=2$ SUSY. This superpotential term is the analogue of the $Q\Phi\widetilde{Q}$ term in $\mathcal{N}=2$ theories with weakly coupled matter. Since the $T_{N,k}$'s also appear as part of $\mathcal{N}=2$ superconformal field theories, they should also contain operators μ_i, $i=1,2,3$ that transform in the adjoint representations of $SU(N)_1$, $SU(N)_2$ and $SU(k)$.

3.4 The Superconformal Index

In this section we review the technology of the superconformal index [27, 28], which we then use as a way of understanding some properties of the $T_{N,k}$ theories. Our main tool is the reduced index for type A theories of class $\mathcal{S}$ found in [29].

3.4.1 Index Basics

The $\mathcal{N}=2$ superconformal index is defined as [27, 28]

$$\mathcal{I} = \mathrm{Tr}\,(-1)^F\, p^{\frac{E-R}{2}+j_1} q^{\frac{E-R}{2}-j_1} u^{-(r+R)}, \tag{3.5}$$

where F is the fermion number, E is the conformal dimension, R is the charge under the Cartan subgroup of the $SU(2)_R$ symmetry, r is the charge under the $U(1)_r$ symmetry, and (j_1, j_2) are the charges under the $SU(2)_1 \times SU(2)_2$ Lorentz group. p, q and u are fugacities which keep track of the quantum numbers for each state in the theory, and the trace is over states on S^3 in the usual radial quantisation. Only states which satisfy the relationship

$$E - 2j_2 - 2R + r = 0 \tag{3.6}$$

contribute to the index.

To help get a feel for this technology, it is useful to compute the "single letter" contributions $f(p,q,u)$. These are the contributions to the index from all single-field operators with arbitrary numbers of derivatives. For vectors and half-hypers, the single-letter partition functions are given by (see e.g. [53])

$$f_{\frac{1}{2}hyper} = \frac{(pq)^{1/4}u^{-1/2} - (pq)^{3/4}u^{1/2}}{(1-p)(1-q)}, \qquad f_{vec} = \frac{(u-u^{-1})(pq)^{1/2} - (p+q) + 2pq}{(1-p)(1-q)}. \tag{3.7}$$

The interesting-looking 2 in the numerator of f_{vec} comes from including a wrong-statistics state with the quantum numbers of a particular equation of motion. Said another way, this term subtracts contributions from states proportional to the quantity which is identified with zero by the equation of motion.

The index in which we will be interested here, the "reduced" index, is obtained by setting $p = q$ and $u = 1$, resulting in

$$\mathcal{I} = \mathrm{Tr}\,(-1)^F\, q^{E-R}. \tag{3.8}$$

It is easy to see that the reduced single-letter partition functions for vectors and hypers are given by

$$f_{\frac{1}{2}hyper,red} = \frac{q^{1/2}}{1-q} \qquad f_{vec,red} = \frac{-2q}{1-q}. \tag{3.9}$$

When flavour symmetries are present, extra fugacities can be introduced to keep track of the charges under the flavour symmetry. This is done in the next section.

3.4.2 The 4d Superconformal Index from q-deformed 2d Yang-Mills

In [29] it was conjectured that for the theories that appear in [26], the reduced index can be obtained using a relation to two-dimensional q-deformed Yang-Mills. Using this relationship, the index for T_N is conjectured to be

$$\mathcal{I}_{T_N}\left(\mathbf{x}_i, q\right)=\frac{\left[\Pi_{i=1}^{\infty}\left(1-q^i\right)\right]^{N-1}\left[\Pi_{i=1}^{3}\eta^{-\frac{1}{2}}\left(\mathbf{x}_i\right)\right]}{\Pi_{\ell=1}^{N-1}\left(1-q^{\ell}\right)^{N-\ell}}\left[\sum_{\mathcal{R}}\frac{1}{\dim_q\mathcal{R}}\chi_{\mathcal{R}}\left(\mathbf{x}_1\right)\chi_{\mathcal{R}}\left(\mathbf{x}_2\right)\chi_{\mathcal{R}}\left(\mathbf{x}_3\right)\right]. \tag{3.10}$$

The fugacities in $\mathcal{I}$ are q (which keeps track of the E and R charges), as well as the vectors $\mathbf{x}_i, i=1,2,3$, which are associated to the three punctures (and keep track of the charges under the flavour symmetry); we will go into greater detail about these below. The sum in (3.10) is over irreducible representations of $SU(N)$, and the q-deformed dimension of a representation is given by

$$\dim_q\mathcal{R}=\prod_{i<j}\frac{\left[\lambda_i-\lambda_j+j-i\right]_q}{\left[j-i\right]_q}, \tag{3.11}$$

where $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{N-1}\geq\lambda_N=0$ are the row lengths of the Young tableau corresponding to the representation, $\mathcal{R}$, and a q-deformed number $[x]_q$ is defined as

$$[x]_q=\frac{q^{-\frac{x}{2}}-q^{\frac{x}{2}}}{q^{-\frac{1}{2}}-q^{\frac{1}{2}}}. \tag{3.12}$$

The characters in equation (3.10) are given by the Schur polynomials:

$$\chi_{\mathcal{R}}\left(\mathbf{x}\right)=\frac{\det\left(x_i^{\lambda_j+N-j}\right)}{\det\left(x_i^{N-j}\right)}, \tag{3.13}$$

where *e.g.* x_i^{N-j} is to be thought of as the entry in the i^{th} row and j^{th} column of a matrix. Finally, the quantity $\eta(\mathbf{x})$ is given by

$$\eta\left(\mathbf{x}\right)=\exp\left\{-2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^n}\chi_{\mathbf{Adj}}\left(\mathbf{x}^n\right)\right\}. \tag{3.14}$$

To get the index for a more general fixture, rather than one with only maximal punctures, we must first associate flavour fugacities to each puncture using the prescription outlined in [29]. The prescription is as follows: Take the Young tableau associated to the puncture, and associate a fugacity to each column of the tableau. For each box in the tableau, associate the fugacity for that column times some power of q. The powers of q should decrease by one down each column and be symmetric about 0; a column with n boxes will begin with the power $q^{(n-1)/2}$. Finally, impose the condition that the product of the quantities associated to each box in the tableau equals 1. This procedure is exemplified in figure 3.6. The conjecture then is that the index for a fixture is given by

aq^2	$bq^{\frac{3}{2}}$	$cq^{\frac{3}{2}}$	$dq^{\frac{3}{2}}$	eq	fq	gq	h	j
aq	$bq^{\frac{1}{2}}$	$cq^{\frac{1}{2}}$	$dq^{\frac{1}{2}}$	e	f	g		
a	$bq^{-\frac{1}{2}}$	$cq^{-\frac{1}{2}}$	$dq^{-\frac{1}{2}}$	eq^{-1}	fq^{-1}	gq^{-1}		
aq^{-1}	$bq^{-\frac{3}{2}}$	$cq^{-\frac{3}{2}}$	$dq^{-\frac{3}{2}}$					
aq^{-2}								

Figure 3.6: An example of the association of flavor fugacities to a Young tableau. The flavour symmetry associated to this puncture is $S\left(U(3)^2 \times U(2) \times U(1)\right) = SU(3)^2 \times SU(2) \times U(1)^3$ and the $S(...)$ constraint imposes $a^5(bcd)^4(efg)^3hj = 1$.

$$\mathcal{I}(\mathbf{x}_i) = \mathcal{N}(q)\left[\prod_{i=1}^{3}\mathcal{A}(\mathbf{x}_i)\right]\sum_{\mathcal{R}}\frac{1}{\dim_q \mathcal{R}}\chi_{\mathcal{R}}(\mathbf{x}_1)\chi_{\mathcal{R}}(\mathbf{x}_2)\chi_{\mathcal{R}}(\mathbf{x}_3), \tag{3.15}$$

where $\mathcal{N}$ and $\mathcal{A}$ are normalisation factors associated to the fixture and punctures, respectively. For the case of a maximal puncture, $\mathcal{A}(\mathbf{x}) = \eta(\mathbf{x})$.

As a warm-up to the $T_{N,k}$, we can expand the expression (3.10) to get some useful information. It is useful to note before we begin that each factor goes to 1 as q goes to zero, so it is easy to read off the low powers of q. First, since

$$\eta^{-\frac{1}{2}}(\mathbf{x}) = 1 + q\,\chi_{\mathbf{Adj}}(\mathbf{x}) + \mathcal{O}\left(q^2\right), \tag{3.16}$$

we see a contribution to $\mathcal{I}_{T_N}$ of the form $q\sum_{i=1}^{3}\chi_{\mathbf{Adj}}(\mathbf{x_i})$. These terms represent the dimension-2 Higgs branch operators μ_i in the adjoint of each $SU(N)$ flavour symmetry of the T_N.

Now consider the terms with $\mathcal{R} = \Box, \overline{\Box}$. Because $\dim_q \overline{\Box} = \dim_q \Box = [N]_q = q^{\frac{-(N-1)}{2}}(1+\mathcal{O}(q))$, there is a term of the form

$$q^{\frac{N-1}{2}}\left[\chi_{\Box}(\mathbf{x}_1)\,\chi_{\Box}(\mathbf{x}_2)\,\chi_{\Box}(\mathbf{x}_3) + \chi_{\overline{\Box}}(\mathbf{x}_1)\,\chi_{\overline{\Box}}(\mathbf{x}_2)\,\chi_{\overline{\Box}}(\mathbf{x}_3)\right]. \tag{3.17}$$

This is the contribution from the dimension-$(N-1)$ operators $\mathcal{O}_{ijk}$ and $\overline{\mathcal{O}}^{ijk}$, which are in the trifundamental and tri-antifundamental representations.

Finally, we look at the term that comes from $\mathcal{R} = \Lambda^l$, the l-index fully antisymmetric representation. Since

$$\frac{1}{\dim_q \Lambda^l} = q^{\frac{l}{2}(N-l)}\left(1 + \mathcal{O}(q)\right), \tag{3.18}$$

there will be a term of the form $q^{\frac{l}{2}(N-l)}\left(\chi_{\Lambda^l}(\mathbf{x_1})\chi_{\Lambda^l}(\mathbf{x_2})\chi_{\Lambda^l}(\mathbf{x_3})\right)$, indicating the presence of dimension-$l(N-l)$ operators in the $\left(\Lambda^l,\Lambda^l,\Lambda^l\right)$ representation. It is interesting to note that at present it is not known how to get these operators with $l\neq 1, N-1$ from duality arguments.

3.4.3 The Superconformal Index for the $T_{N,k}$

We now look at calculating the reduced superconformal index for the $T_{N,k}$ theory. The $T_{N,k}$ theory has two maximal punctures and one puncture with pole structure $\{1,2,...,k-1,k,k,...k\}$, so the superconformal index is

$$\mathcal{I}_{T_{N,k}}(\mathbf{x}_i)=\mathcal{N}_{T_{N,k}}(q)\left[\prod_{i=1}^{2}\eta^{-\frac{1}{2}}(\mathbf{x}_i)\right]\mathcal{A}(\mathbf{x}_3)\left[\sum_{\mathcal{R}}\frac{\chi_{\mathcal{R}}(\mathbf{x}_1)\chi_{\mathcal{R}}(\mathbf{x}_2)\chi_{\mathcal{R}}(\mathbf{x_3})}{\dim_q\mathcal{R}}\right].$$

The Young tableau for the non-maximal puncture has one column of height $N-k$ and k columns of height one, giving the flavour fugacities

$$\mathbf{x_3}=\left(aq^{\frac{N-k-1}{2}},...,aq^{-\frac{N-k-1}{2}},b_1a^{\frac{k-N}{k}},b_2a^{\frac{k-N}{k}},...,b_{k-1}a^{\frac{k-N}{k}},\left[\Pi_{i=1}^{k-1}b_i\right]^{-1}a^{\frac{k-N}{k}}\right). \quad (3.19)$$

We first look at the terms in the last factor with $\mathcal{R}=\square,\overline{\square}$. Eq. (3.13) gives us $\chi_{\square}(\mathbf{x})=\sum_i x_i$, so the characters of the third puncture are given by

$$\chi_{\square}(\mathbf{x}_3)=aq^{-\frac{N-k-1}{2}}+...+aq^{\frac{N-k-1}{2}}+a^{\frac{k-N}{k}}\chi_{\square}(\mathbf{b}), \quad (3.20)$$

with $\chi_{\overline{\square}}$ given by taking $a\to 1/a$ and $\square\to\overline{\square}$. $\chi_{\square}(\mathbf{b})$ and $\chi_{\overline{\square}}(\mathbf{b})$ are the characters of the fundamental and anti-fundamental representations of $SU(k)$ in terms of the flavour fugacities $b_1,...,b_{k-1}$. Multiplying by the factors from the maximal punctures, $q^{\frac{1}{2}(N-1)}\chi_{\square}(\mathbf{x}_1)\chi_{\square}(\mathbf{x}_2)$ and $q^{\frac{1}{2}(N-1)}\chi_{\overline{\square}}(\mathbf{x}_1)\chi_{\overline{\square}}(\mathbf{x}_2)$ respectively, we see the presence of operators given in table 3.1. In addition to the (perhaps expected) presence of trifundamental and tri-antifundamental operators, it is interesting to note the presence of bifundamentals of various dimensions increasing in increments of two including the dimension k bifundamental operators found in the last section.

No of Operators	Representation	Dimension
$N-k$	$(\square,\square,\mathbf{1})_1$	$k,k+2,k+4,...,2N-k-2$
$N-k$	$(\overline{\square},\overline{\square},\mathbf{1})_{-1}$	$k,k+2,k+4,...,2N-k-2$
1	$(\square,\square,\square)_{\frac{k-N}{k}}$	$N-1$
1	$(\overline{\square},\overline{\square},\overline{\square})_{\frac{N-k}{k}}$	$N-1$

Table 3.1: The flavor symmetry representations (under $SU(N)^2\times SU(k)\times U(1)$) and dimensions of operators for the $T_{N,k}$.

Note that the number of operators coming from the $\mathcal{R}=\square$ or $\mathcal{R}=\overline{\square}$ terms is $N^2(N-$

$k+k) = N^3$, which is the same counting as in the analogous terms for the T_N. This has to be the case since $\chi(\mathbf{x}_3)$ has the same number of terms regardless of the puncture. Moreover, when $k = N-1$, the dimension-$(N-1)$ bifundamental and trifundamental operators combine to give us the trifundamental operator of the T_N, as expected.

It is also interesting to consider the operators transforming in the various ℓ-index antisymmetric representations Λ^ℓ. Using equation (3.13) one can see that the characters of these representations are given by

$$\chi_{\Lambda^l}(\mathbf{x}) = \sum_{\substack{i_1=1,\\ i_2>i_1,\ldots,i_l>i_{l-1}}}^{N} x_{i_1}x_{i_2}\ldots x_{i_l}, \tag{3.21}$$

and so for the third puncture we have

$$\chi_{\Lambda^l}(\mathbf{x_3}) = \sum_{l'=0}^{\min(l,k)} a^{l-l'\left(\frac{N}{k}\right)} \left[\sum_{\substack{i_1=-\frac{N-k-1}{2},\\ i_2>i_1,\ldots,i_{l-l'}>i_{l-l'-1}}}^{\frac{N-k-1}{2}} q^{i_1}q^{i_2}\ldots q^{i_{l-l'}}\right]\chi_{\Lambda^{l'}}(\mathbf{b}). \tag{3.22}$$

We find that the operators coming from the $\mathcal{R}=\Lambda^l$ term are those given in table 3.2. Again we see that the number of operators coming from the $\mathcal{R}=\Lambda^l$ term is equal to that of the T_N, which we can see via Vandermonde's identity:

$$\sum_{l'=0}^{\min(l,k)} \binom{N-k}{l-l'}\binom{k}{l'} = \binom{N}{l}. \tag{3.23}$$

No of Operators	Representation	Dimension
$\binom{N-k}{l}$	$\left(\Lambda^l,\Lambda^l,\mathbf{1}\right)_l$	$[lk, 2lN-kl-2l^2]$
$\binom{N-k}{l-1}$	$\left(\Lambda^l,\Lambda^l,\square\right)_{l-\left(\frac{N}{k}\right)}$	$[N+kl-2l-k-1,$ $2lN-N-2l^2-kl+2l+k-1]$
$\vdots$	$\vdots$	$\vdots$
$\binom{N-k}{l-l'}$	$\left(\Lambda^l,\Lambda^l,\Lambda^{l'}\right)_{l-l'\left(\frac{N}{k}\right)}$	$[lk+l'N-l'k-2ll'+l'^2,$ $(k+2l)(l'-l)+N(2l-l')-l'^2]$

Table 3.2: This table gives the flavour symmetry representations (under $SU(N)^2\times SU(k)\times U(1)$) and range of dimensions of operators of the $T_{N,k}$. l' will stop at l or k, whichever is less.

3.5 New SCFTs and Flows

In this section we use the $T_{N,k}$ theories to construct new $\mathcal{N}=1$ SCFTs, and describe flows between these theories. The analysis in this section extends the work done in [18] and answers an open question about flows that appeared to violate the a-theorem. We note here that the evidence presented in this and the following section is necessary but not sufficient for the theories in question to be SCFTs. Although some of the theories we will build have obvious problems such as unitarity violations, it is possible that even the ones that do not appear to be problematic do not actually dynamically reach a conformal fixed point. To determine without question whether or not the theories we consider are conformal would require stronger evidence, such as an AdS dual. Nevertheless, we believe the evidence presented here is suggestive that many of these theories are SCFTs.

3.5.1 S_ℓ Theories With T_N's

An S_ℓ theory, first analysed in [18], is an $\mathcal{N}=1$ $SU(N)^{\ell+1}$ gauge theory with ℓ bifundamental hypermultiplets, two T_N's, and an $SU(N)^4\times U(1)\times U(1)_R$ global anomaly-free symmetry. The theory is represented by the generalised quiver shown in figure 3.7. Since we are now dealing with $\mathcal{N}=1$ theories, in this section circles will correspond to $\mathcal{N}=$

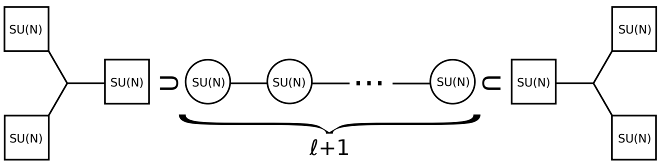

Figure 3.7: The S_ℓ quiver.

1 vector multiplets. Lines will still correspond to bifundamental $\mathcal{N}=2$ hypermultiplets, or in $\mathcal{N}=1$ language, two chiral multiplets in the $(\square,\overline{\square})$ and $(\overline{\square},\square)$ representations.

A useful global symmetry is

$$R_0 = R_{\mathcal{N}=1} + \frac{1}{6}J = \frac{1}{2}R_{\mathcal{N}=2} + I_3, \tag{3.24}$$

which is the R-symmetry preserved when flowing to the S_ℓ theory by giving masses to adjoint chiral superfields in vector multiplets in the analogous $\mathcal{N}=2$ theory. The $U(1)$ global symmetries $R_{\mathcal{N}=2}, R_{\mathcal{N}=1}, J$, and I_3 are detailed in [18]. Here we only note that $R_{\mathcal{N}=2}$ and I_3 are the charges under the $U(1)_R\times U(1)_r$ which descends from the $U(1)_R\times SU(2)_r$ $\mathcal{N}=2$ R-symmetry; $R_{\mathcal{N}=1}$ and J are just particular linear combinations. Additionally, each of the bifundamentals comes with a $U(1)$ which we call F_i, normalised as $F_i(Q_j)=F_i(\widetilde{Q}_j)=\delta_{ij}$ where $Q_j,\widetilde{Q}_j$ are the j^{th} bifundamentals. These F_i are individually anomalous but can be combined into the anomaly-free global symmetry

$$\mathcal{F} = J_1 + \sum_{i=1}^{\ell}(-1)^{i-1}F_i + (-1)^{\ell-1}J_2, \tag{3.25}$$

where $J_{1,2}$ are global symmetries under which only the T_N theories are charged. When ℓ is even, $\mathrm{Tr}\,\mathcal{F}=0$, so $\mathcal{F}$ will not mix with the R-symmetry [20], which is therefore R_0.

When ℓ is odd, $\mathrm{Tr}\,\mathcal{F}\neq 0$, so we must use a-maximization [20] to determine the IR R-symmetry. In other words, we must find the value of α that maximises

$$a_{trial}(\alpha) = 3\,\mathrm{Tr}\,R_{trial}^3 - \mathrm{Tr}\,R_{trial}, \tag{3.26}$$

where $R_{trial}(\alpha) = R_0 + \alpha\mathcal{F}$. This was done in [18], with the result that

$$\widehat{\alpha} = \frac{A-\sqrt{B}}{C}, \tag{3.27}$$

where

$$\begin{aligned}
A &= 4N^3 + 3\ell N^2 - 4N,\\
B &= 64N^6 + 8\,(3\ell-25)\,N^5 + 3\left(3\ell^2+41\right)N^4 - 24\,(\ell-9)\,N^3 - 208N^2 - 64N + 64,\\
C &= 6\left(4N^3 - 11N^2 + 8\right).
\end{aligned}$$

3.5.2 S_ℓ Theories With $T_{N,k}$'s

We now look at the S_ℓ theory as in the last section but now with $T_{N,k}$'s and an $SU(N)\subset SU(N)^2\times SU(k)\times U(1)$ gauged at each end of the quiver. We again find that there is an anomaly-free R-symmetry as in equation (3.24) and an anomaly-free $U(1)$ flavor symmetry as in equation (3.25). As before, the case with even ℓ is trivial, and the R-symmetry is R_0. However, for ℓ odd, we must use a-maximization.

If we perform a-maximization then we find that the value of α that maximises a_{trial} is

$$\widehat{\alpha} = \frac{A+\sqrt{B}}{C}, \tag{3.28}$$

where

$$\begin{aligned}
A &= -(3\ell+6k)N^2 + 2k^3 - 2k,\\
B &= N^4\left(144k^2 + 36k\ell - 204k + 9\ell^2 + 91\right)\\
&\quad + N^2\left(-96k^4 - 12k^3\ell - 28k^3 + 80k^2 + 12k\ell + 204k - 160\right)\\
&\quad + 16k^6 + 32k^5 + 16k^4 - 64k^3 - 64k^2 + 64,\\
C &= 6\left((7-6k)N^2 + 2k^3 + 4k^2 + 2k - 8\right).
\end{aligned}$$

$\widehat{\alpha}$, which is plotted in figure 3.8, seems to be negative for all values of ℓ, N and k and approaches $\frac{-6k-3\ell+\sqrt{144k^2-204k+36k\ell+9\ell^2+91}}{6(7-6k)}$ at large N.

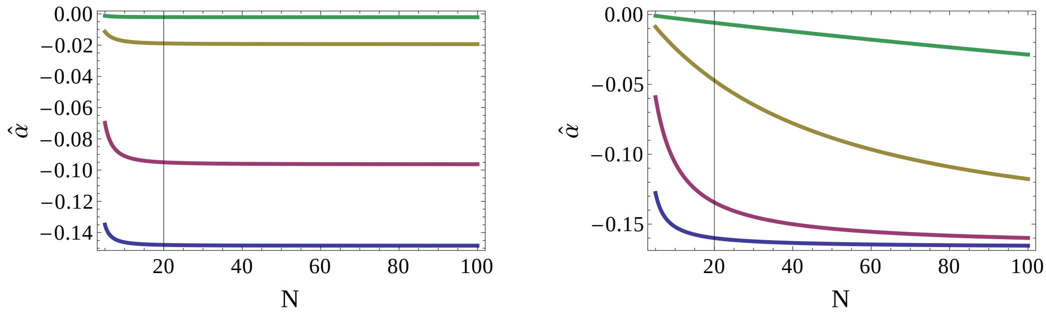

Figure 3.8: The value of α that maximises a_{trial} for the S_ℓ theory with two $T_{N,k}$'s. α is plotted against N for $k=5$ (left), and $k=N-2$ (right) each with $\ell=1$ (blue), 11 (purple), 101 (yellow), 1001 (green).

In order to verify that there are no gauge-invariant operators in this theory that violate the unitarity bound $R \geq \frac{2}{3}$ we note that α never goes below $-\frac{1}{6}$ for any k and ℓ. One can then easily verify using the results of table 3.3 that indeed no gauge-invariant operators violate unitarity.

Operator	R-charge
Q_i	$\frac{1}{2}+(-1)^{i-1}\alpha$
$\widetilde{Q}^i$	$\frac{1}{2}+(-1)^{i-1}\alpha$
μ	$1-2\alpha$
$\mathcal{O}_H$	$\left(\frac{1}{2}-\alpha\right)\Delta_{UV}$
u_n	$(1+2\alpha)\,n$

Table 3.3: Operator dimensions for the S_ℓ theory with $T_{N,k}$'s. $\mathcal{O}_H$ is any of the Higgs branch operators in table 3.2 and Δ_{UV} is the operator's dimension given in the same table. u_n are the Coulomb branch operators.

We can also ask what happens when we construct the S_ℓ theory with two different $T_{N,k}$'s at either end of the quiver, *i.e.*, T_{N,k_1} and T_{N,k_2}. The behaviour is qualitatively similar to when $k_1=k_2$, and we have included the result in the appendix. For now, we merely note that no gauge-invariant operators violate the unitarity bound, so these theories do not appear to be problematic.

3.5.3 Other $\mathcal{N}=1$ Theories With $T_{N,k}$'s

In [18], the authors additionally studied two other theories formed from T_N's and Lagrangian matter, namely the $S_\ell^\square$ and the S_ℓ°. We now wish to construct analogues of the theories using $T_{N,k}$'s instead of T_N's. The generalised quiver diagrams for these theories are given in figures 3.9 and 3.10.

The extension of the $S_\ell^\square$ theory is straightforward, since it is a special case of the theories at the end of the previous section. Because the $T_{N,1}$ is a set of N^2 free hypermultiplets, the theories at the end of the previous section with $k_2=1$ are the $S_\ell^\square$ theory.

For the S_ℓ° theories, we use the same R_0 symmetry as in the previous section. In order for R_0 to be anomaly-free we require that $R_0(\Phi)=\frac{1}{2}$, where Φ is the adjoint chiral superfield. The extra adjoint chiral superfield Φ comes with a $U(1)$ flavour symmetry F_a

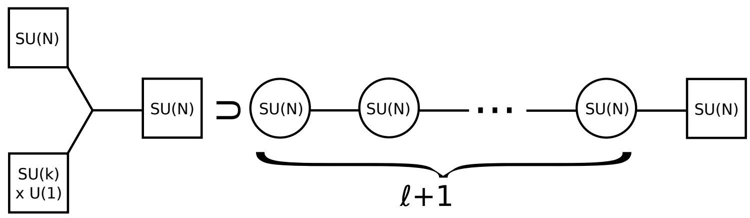

Figure 3.9: The generalised quiver diagram for an $S_\ell^\square$ theory.

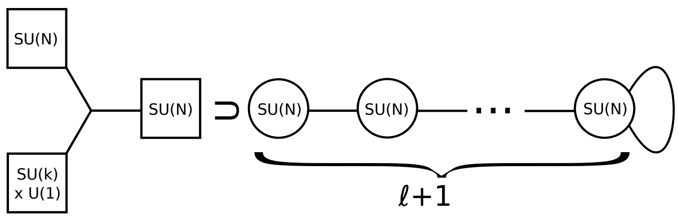

Figure 3.10: The generalised quiver diagram for an S_ℓ° theory. The loop denotes a chiral superfield in the adjoint representation.

which we normalise so that $F_a(\Phi) = 1$. The only anomaly-free $U(1)$ symmetry is

$$\mathcal{F} = J_1 + \sum_{i=1}^{\ell}(-1)^{i-1}F_i + (-1)^\ell F_a \tag{3.29}$$

Again we can get the IR R-symmetry by maximising $a_{trial}(\alpha) = 3\,\mathrm{Tr}\,R_{trial}^3 - \mathrm{Tr}\,R_{trial}$ with respect to α, where $R_{trial} = R_0 + \alpha\mathcal{F}$. The answer is unwieldy, so we merely note that α does not seem to drop below $-\frac{1}{6}$ for any ℓ, k and consequently there are no unitarity bound violations for the same reasons as for the S_ℓ theories. Thus, the theories in this subsection are likely to be good SCFTs.

It is interesting to note that when we add the superpotential term $Q_\ell\Phi\widetilde{Q}_\ell$ to the theory, some of the operators violate unitarity. In the theory without this superpotential term the R-charge of the $Q_\ell\Phi\widetilde{Q}_\ell$ operator is $R(Q_\ell\Phi\widetilde{Q}_\ell) = \frac{3}{2} - (-1)^\ell\,\widehat{\alpha}$, where $\widehat{\alpha}$ is the value of α that maximises a_{trial}. There are also operators in the theory $\mathrm{Tr}(\Phi^n)$ which have R-charge $R(\Phi^n) = n\left(\frac{1}{2} + (-1)^\ell\widehat{\alpha}\right)$. In the theory with the superpotential term turned on a-maximization is not needed because the R charge of the $Q_\ell\Phi\widetilde{Q}_\ell$ term is fixed to equal 2. This effectively sets the value of $\widehat{\alpha}$ so that $(-1)^\ell\widehat{\alpha} = -\frac{1}{2}$. This means that the R-charge of the $\mathrm{Tr}(\Phi^n)$ operators will be zero. Thus, these theories with the superpotential term turned on appear to be problematic, and are likely not SCFTs.

3.5.4 Flows From Higgsing

We now look at what happens when we take an S_ℓ theory with $T_{N,k}$'s and give a vev to the k-th hypermultiplet. In [18] it was argued that the theory that emerges in the IR is the $S_{\ell-1}$ theory with a chiral superfield Φ in the adjoint representation of the $(k-1)$-th gauge group [5]. This is represented by the quivers in figure 3.11.

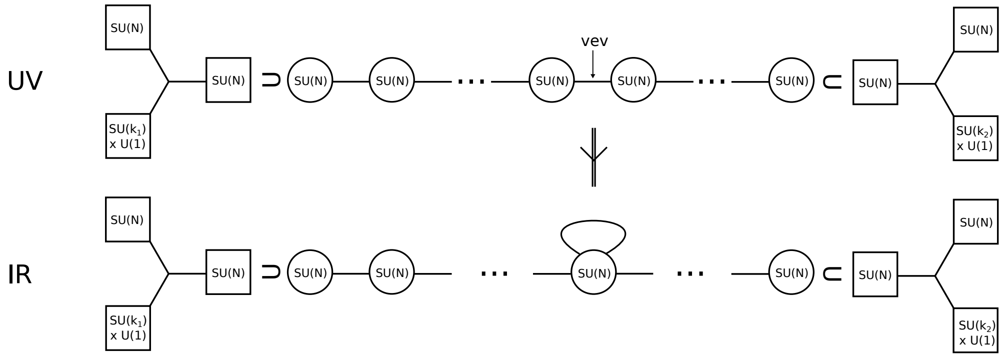

Figure 3.11: If we give a vev to the k-th hypermultiplet in the S_ℓ theory (above) then this induces a flow to the $S_{\ell-1}$ theory with an chiral superfield in the adjoint representation of the $(k-1)$-th gauge group.

We note that in the UV theory there are the marginal operators $Q_{k-1}\widetilde{Q}_{k-1}Q_k\widetilde{Q}_k$ and $Q_k\widetilde{Q}_kQ_{k+1}\widetilde{Q}_{k+1}$, so we must in general consider these terms to be turned on. After Higgsing, these terms become $Q_{k-1}\Phi\widetilde{Q}_{k-1}$ and $Q_{k+1}\Phi\widetilde{Q}_{k+1}$ respectively, where Φ is the adjoint chiral superfield. These superpotential terms were not taken into consideration in [18], and since they are allowed by all symmetries, should in general be included.[6]

Including such terms results in the one-parameter family of R-symmetries

$$R_{trial} = R_0 + \alpha\mathcal{F}, \tag{3.30}$$

where the additional anomaly-free $U(1)$ symmetry is

$$\begin{aligned}\mathcal{F} =& J_1 + F_1 - F_2 + ... + (-1)^k F_{k-1} + 2(-1)^{k+1}F_a + (-1)^{k+2}F_{k+1} + ...\\ &+ (-1)^\ell F_{\ell-1} + (-1)^\ell J_2.\end{aligned} \tag{3.31}$$

In this formula F_a is the additional $U(1)$ symmetry that comes with the adjoint chiral superfield, which we normalise as $F_a(\Phi) = 1$. We can then use a-maximization to find the value of α that maximises a; this result is again in the appendix.

If we then calculate $a_{UV} - a_{IR}$ then we see that there are no a-theorem violations for this flow. The value of $a_{UV} - a_{IR}$ for even and odd ℓ is plotted against N in figure 3.12. We can repeat this analysis for the S_ℓ with two general T_{N,k_1}, T_{N,k_2} and we find that there are no a-theorem violations for any of these flows. Although not in and of itself conclusive, the fact that none of these flows violates the a-theorem lends credence to the existence of the IR theories as interacting conformal points. This is perhaps not surprising, since many examples of such quivers which mix $\mathcal{N} = 1$ and $\mathcal{N} = 2$ vector multiplets are now known to be SCFTs, though these were not known at the time of the original work [18].

[5] The authors of [18] only considered the S_ℓ theory with T_N's but the argument still holds.
[6] A similar point was discussed in [34].

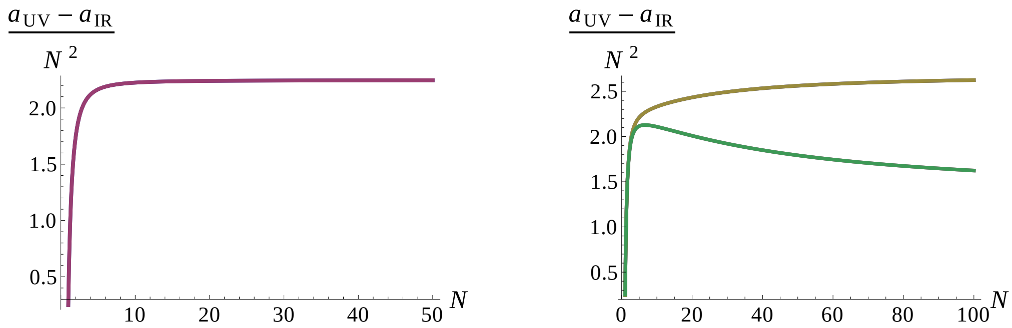

Figure 3.12: Left: $a_{UV} - a_{IR}$, which is independent of k, plotted for even ℓ. Right: $a_{UV} - a_{IR}$ plotted for ℓ odd and k even (bottom) and odd (top).

3.5.5 Linear Deformations of the $T_{N,k}$

In this section we look at what happens when we deform a $T_{N,k}$ theory with an operator of the form $\mathrm{Tr}(\phi\mu)$, where ϕ is a constant adjoint-valued matrix. This in general breaks the flavour symmetry of the $T_{N,k}$ and drives a flow to a new theory in the IR. We can use the methods of [54] to determine the R-symmetry in the IR.

For simplicity, as was done in [54], we assume that the matrix ϕ takes block diagonal form, $\phi = \oplus_a \phi^{(a)}$, where each $\phi^{(a)}$ is an $n_a \times n_a$ upper-diagonal matrix; this breaks the theory to $\mathcal{N} = 1$. Then there is an $SU(2)$ subalgebra of the original flavour symmetry associated to each $\phi^{(a)}$, where $\phi^{(a)}$ is in the spin-$\frac{1}{2}(n_a - 1)$ representation. As discussed in [54], the entries of $\phi^{(a)}$ along the first superdiagonal are the most relevant and drive the flow, so we further assume that each $\phi^{(a)}$ is a nilpotent Jordan block.

The IR R-symmetry then is given by

$$R_{IR} = \frac{t}{2} R_{\mathcal{N}=2} + (2-t) I_3 - t T_3, \tag{3.32}$$

where $T_3 = \sum_a T_3^{(a)}$ and $T_3^{(a)}$ is the generator of the Cartan subalgebra of the $SU(2)$ flavor symmetry associated to $\phi^{(a)}$; t is determined by a-maximization to be

$$t = \frac{4}{3} \times \frac{8a_{T_{N,k}} - 4c_{T_{N,k}} - \sqrt{4c_{T_{N,k}}^2 + (4a_{T_{N,k}} - c_{T_{N,k}})k_G r}}{16a_{T_{N,k}} - 12c_{T_{N,k}} - k_G r}, \tag{3.33}$$

where k_G is the central charge of the flavour symmetry under which the μ we are deforming with transforms; $r = 2\,\mathrm{Tr}(T_3 T_3)$ measures the sizes of the $\phi^{(a)}$ blocks. We will look only at the case where ϕ has just a single $n \times n$ upper-diagonal block so that $r = \frac{n^3-n}{6}$.

We can calculate the central charge a for this theory in the IR and compare it to a_{UV} given in eq.(3.2). When we do this, we see that $a_{UV}(N,k) - a_{IR}(N,k,n)$ never drops below zero, as dictated by the a-theorem, and is monotonically increasing in N, k and n. This makes intuitive sense, because increasing N or k corresponds to adding degrees of freedom to the UV theory, and increasing n corresponds to integrating out a larger proportion of the IR degrees of freedom.

We can also ask what the operator dimensions are in these theories. First we look at the Coulomb branch operators, which have dimension equal to $\frac{3}{2}t\Delta_{UV}$. Since the lowest lying operators have dimension $\Delta_{UV} = 3$, to check if unitarity bounds are violated, it suffices to check if t drops below $\frac{2}{9}$. It is easy to show that this happens for many values of N and k provided n is large enough. Thus it seems that these are likely not good SCFTs in general, although it is possible that there is some interesting reason for critical values of n. In the absence of an understanding of why this transition should happen, it seems most reasonable to conclude that none of these theories are conformal.

We now look at the dimension of the μ operator. Because we deformed by a $\mathrm{Tr}(\phi\mu)$ operator and identified a $U(1)$ symmetry to ϕ (or equivalently μ) we see that μ splits up into many operators with different T_3 charge. The dimensions of these operators are $\frac{3}{2}\left(2 - t(1+T_3)\right)$. It is easy to verify that many of these operators violate the unitarity bound, $\Delta \geq 1$, for many values of N, k and n.

For the Higgs branch operators that we found using the superconformal index (i.e. those given in table 3.2) the operator dimensions are $\frac{3}{2}\left(\Delta_{UV}\left(1-\frac{t}{2}\right) - tT_3\right)$ and again we can see many cases of unitarity violations.

We can also ask what happens when we deform by more than one of the μ operators (i.e. deform by $\mathrm{Tr}\left(\phi_1\mu_1 + \phi_2\mu_2 + \phi_3\mu_3\right)$). Using the same reasoning as [54] and above we find that the R-charge is

$$R_{IR} = \frac{t}{2}R_{\mathcal{N}=2} + (2-t)I_3 - t(T_3^{(1)} + T_3^{(2)} + T_3^{(3)}), \tag{3.34}$$

where now there is a T_3 assigned to each ϕ (or equivalently each μ). The value of t is again determined by a-maximization to be the same as that given in (3.33) with the replacement $k_G r \to \sum_i^3 k_G^{(i)} r^{(i)}$. The analysis is qualitatively the same as has been done already.

It is worth noting that these unitarity violations appear to persist even for $k = 1$ if one naïvely uses eq. (3.33). However, there the theory consists of free hypermultiplets, and the superpotential deformation $W = \mathrm{Tr}(\phi\mu)$ gives these a mass, so no unitarity problems should occur. In this case, the enhanced symmetry of the free hypers makes a-maximization unnecessary, and the theory retains $\mathcal{N} = 2$ SUSY, so eq. (3.33) is not applicable.

We also note that for theories exhibiting unitarity violating operators, this violation could be remedied by an emergent IR symmetry which would require a-maximization to be done again, as in [55]. Thus, the apparent violation of unitarity by a few operators does not necessarily mean that the theory is not conformal.

3.6 Theories Without Non-Abelian Flavor Symmetries

In this section we construct an interesting family of new SCFTs using the $T_{N,k}$ theories as building blocks. This approach mirrors that of [30, 4, 36] which used T_N's coupled by gauging $SU(N)_{diag} \subset SU(N)_1 \times SU(N)_2$, where $SU(N)_1$ and $SU(N)_2$ can belong to different T_N's. The introduction of $T_{N,k}$'s creates significant differences, and the classification of the allowed theories is significantly more complex than that of the T_N quivers. In this section we examine various aspects of these theories and in particular find a puzzle relating to the dimensions of the conformal manifolds.

3.6.1 B^{3}W Theories

We begin with a brief review of the theories in [30, 36], which we refer to as the B^{3}W theories. These theories are constructed by taking a collection of T_N theories and gauging an $SU(N)_{diag} \subset SU(N) \times SU(N)$. The vector multiplet associated to $SU(N)_{diag}$ can be either $\mathcal{N} = 1$ or $\mathcal{N} = 2$. These theories can usefully be pictured by generalised quivers, where we use the convention that white circles are $\mathcal{N} = 2$ multiplets, while black circles are $\mathcal{N} = 1$. In addition to a $U(1)$ R-symmetry, these theories also possess exactly one anomaly-free non-R $U(1)$ global symmetry $\mathcal{F}$. When $\mathrm{Tr}\,\mathcal{F} \neq 0$, this can mix with the R-symmetry, and a-maximization is necessary. A useful choice of R-symmetry is

$$R_0 = R_{\mathcal{N}=1} + \frac{1}{6}\sum_i J_i + \frac{1}{3}\sum_A F_A, \tag{3.35}$$

and the non-R anomaly-free $U(1)$ symmetry can be taken to be

$$\mathcal{F} = \sum_i \sigma_i J_i + 2\sum_A \sigma_A F_A, \tag{3.36}$$

where F_A is the $U(1)$ symmetry associated to the adjoint chiral superfield living in the A-th $\mathcal{N} = 2$ vector multiplet, normalised so that $F_A(\Phi_B) = \delta_{AB}$, and assigning a sign $\sigma_i = \pm 1$ to each T_N. The B^{3}W theories follow the rule that T_N's of opposite sign must be connected by a shaded node and T_N's of the same sign must be connected by an unshaded node. One then also assigns a sign $\sigma_A = \pm 1$ to each $\mathcal{N} = 2$ vector multiplet, depending on whether it connects two T_N's of positive or negative sign, respectively. Two examples of these theories are given in figure 3.13.

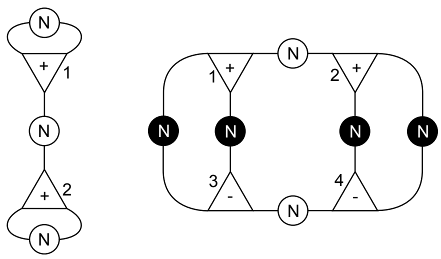

Figure 3.13: Two examples of B^{3}W theories: one with genus two (left) and one with genus three (right).

The supergravity duals of these theories were also found in [30, 36]. In this construction, which generalises the famous Maldacena-Nuñez result [25], the authors found the near-horizon geometries for an infinite family of $\mathcal{N} = 1$ theories that come from M5-branes wrapping a Riemann surface. The theories are specified by two integer parameters p and q, which for $p, q \geq 0$ are dual to the above quivers. In the UV, this geometry can also be thought of as M5-branes wrapping a Riemann surface Σ_g inside a Calabi-Yau, such that the total space is a decomposable line bundle $\mathcal{L}_1 \oplus \mathcal{L}_2 \to \Sigma_g$, with p and q being the Chern numbers of each factor in the bundle. The Calabi-Yau condition then requires $p + q = 2g - 2$. In the dual field theory, p and q have the interpretation of being the number of σ_i of each sign. One check of the duality of the two sides of the AdS/CFT correspondence is the leading-order agreement between the central charges computed on either side; the next-order agreement was found in [56].

Another check of the correspondence is given by the dimensions of the conformal manifolds for these theories, which is $4g - 3$. This quantity can be easily computed in the field theory via either Leigh-Strassler [23] or the technique of [24]. Geometrically, the marginal deformations can be thought of as the $3g - 3$ complex structure deformations of the Riemann surface along with the g allowed shifts of the Wilson lines around each cycle by a flat connection, for a total of $4g - 3$.

3.6.2 Our Setup

Here, we will use $T_{N,k}$'s to construct analogues of the B^{3}W theories. This change leads to some profound differences; in particular, we no longer have a known AdS solution that we can use to check our answers. Nevertheless, we will provide some evidence that these constructions lead to interesting new SCFTs in the IR.

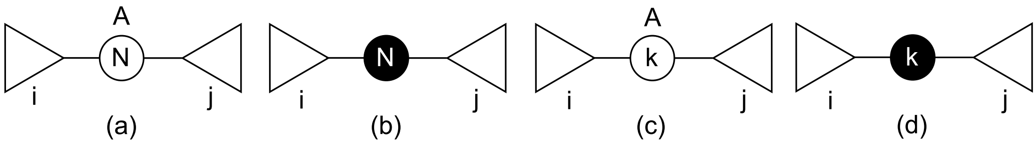

Figure 3.14: The four different ways of coupling $T_{N,k}$'s to vector multiplets. A triangle denotes a $T_{N,k}$, a shaded circle with an n denotes an $\mathcal{N} = 1$ $SU(n)$ vector multiplet and an unshaded circle with an n denotes an $\mathcal{N} = 2$ $SU(n)$ vector multiplet.

The first thing we must determine is how to couple $T_{N,k}$'s to vector multiplets by gauging diagonal subgroups of flavour symmetries. There are four different ways of doing this, as shown in figure 3.14. The first two gaugings, a and b, are the same as we had with the B^{3}W theories, and gaugings c and d are new. The first of these new gaugings is an $SU(k)$ gauging using an $\mathcal{N} = 2$ vector multiplet and the second is an $SU(k)$ gauging with an $\mathcal{N} = 1$ vector multiplet. The 1-loop beta function coefficients b_0 for the gauge couplings for each of these gauge groups are:

$SU(N)$

(a) $\mathcal{N}=2 \Rightarrow -b_0 = 2T(G) - 2\frac{1}{2}k_{SU(N)} = 2N - 2N = 0$

(b) $\mathcal{N}=1 \Rightarrow -b_0 = 3T(G) - 2\frac{1}{2}k_{SU(N)} = 3N - 2N = N$

$SU(k)$

(c) $\mathcal{N}=2 \Rightarrow -b_0 = 2T(G) - 2\frac{1}{2}k_{SU(k)} = 2(k) - 2(k+1) = -2$

(d) $\mathcal{N}=1 \Rightarrow -b_0 = 3T(G) - 2\frac{1}{2}k_{SU(k)} = 3(k) - 2(k+1) = k-2$

Of the two new gaugings, only the $\mathcal{N}=1$ $SU(k)$ will become strongly coupled in the IR. The $\mathcal{N}=2$ gauging goes free in the IR. In the theories we wish to construct we will not be interested in this type of gauging, so we will only consider gaugings a, b, and d.

The theories we study here will, as before, be constructed by taking an even number of $T_{N,k}$'s and gauging diagonal subgroups until there is no non-Abelian flavour symmetry left. Note that one difference with the B^{3}W theories is that here we only gauge $SU(N)$ or $SU(k)$ inside $SU(N)^2 \times SU(k) \times U(1)$. This means that the theories that we construct will have a residual $U(1)^n$, where n is the number of $T_{N,k}$'s, in addition to any additional anomaly-free $U(1)$ which is a linear combination of J_i's and F_A's. Each of these residual $U(1)$ factors is trace-free, so they will not mix with the IR R-symmetry.

These theories have an IR R-symmetry of the form

$$R_0 = R_{\mathcal{N}=1} + \sum_i \alpha_i J_i + \sum_A \beta_A F_A. \tag{3.37}$$

The anomaly-free condition Tr $R_0 T^a T^b = 0$ gives us constraints on the constants α_i, β_A, and there are additional constraints from enforcing that superpotential terms $\mu\Phi$, which are necessary for an $\mathcal{N}=2$ gauging, have R-charge two. For gaugings of type a, b and d in figure 3.14 these constraints are as follows:

$$\begin{aligned} SU(N)\ \ \mathcal{N}=2 &\Rightarrow \alpha_i = \alpha_j = \frac{1}{2}\beta_A, \\ \mathcal{N}=1 &\Rightarrow \alpha_i + \alpha_j = \frac{1}{3}, \\ SU(k)\ \ \mathcal{N}=1 &\Rightarrow \alpha_i + \alpha_j = \frac{k}{k+1} - \frac{2}{3}. \end{aligned} \tag{3.38}$$

Any anomaly-free additional $U(1)$ current will be of the form

$$\mathcal{F} = \sum_{i=1} \mu_i J_i + \sum_A \nu_A F_A, \tag{3.39}$$

and the anomaly-free constraint $\mathrm{Tr}\,\mathcal{F}T^aT^b = 0$, along with the constraint $\mathcal{F}(\mu\Phi) = 0$, imposes the following:

$$\begin{aligned}
&\mathcal{N} = 2 \Rightarrow \mu_i = \mu_j = \frac{1}{2}\nu_A, \\
&\mathcal{N} = 1 \Rightarrow \mu_i + \mu_j = 0.
\end{aligned} \tag{3.40}$$

An interesting difference between the original B^{3}W theories and the theories we wish to construct here is that the additional anomaly-free $U(1)$ symmetry will always be traceless. One way to see this is to note that because each $T_{N,k}$ has only one $SU(k)$ factor, the $T_{N,k}$'s come in pairs connected by $\mathcal{N} = 1$ $SU(k)$ vector multiplets. For a pair that connects the i^{th} and j^{th} $T_{N,k}$, the anomaly-free constraint on $\mathcal{F}$ is that $\mu_i = -\mu_j$. This means that the first term in (3.39) vanishes. One can also show that all $\mathcal{N} = 2$ vector multiplets either: (a) come in pairs with cancelling contributions to $\mathcal{F}$ (i.e. $\nu_A = \nu_B$), or; (b) have $\nu_A = 0$. This means that for the theories we construct here we will never need to use a-maximization to determine the IR R-symmetry.

3.6.3 A Subclass of Theories

We first consider a subclass of theories that are constructed in the UV from $T_{N,k}$'s, $\mathcal{N} = 1$ $SU(k)$ vector multiplets, and $\mathcal{N} = 2$ $SU(N)$ vector multiplets. For the moment, we do not include $\mathcal{N} = 1$ $SU(N)$ vectors. An example of one of these theories is given by the quiver in figure 3.15.

Using the rules in (3.38) and symmetry of the quiver diagram we find that the IR R-symmetry is given by

$$\begin{aligned}
R_{IR} =& R_{\mathcal{N}=1} + \frac{1}{2}\left(\frac{k}{k+1} - \frac{2}{3}\right)\sum_i J_i \\
&+ \left(\frac{k}{k+1} - \frac{2}{3}\right)\sum_A F_A.
\end{aligned} \tag{3.41}$$

Operator	R-charge
μ_i	$2 - \frac{k}{k+1}$
$u_n^{(i)}$	$n\left(\frac{k}{k+1}\right)$
$\mathcal{O}_H^i$	$\Delta_{UV}\left(1 - \frac{1}{2}\left(\frac{k}{k+1}\right)\right)$
Φ_A^n	$n\left(\frac{k}{k+1}\right)$

Table 3.4: Some operators of the IR theory with R-charges. μ_i are the μ operators and $u_n^{(i)}$ ($n \geq 3$) are the Coulomb branch operators for the i^{th} $T_{N,k}$. $\mathcal{O}_H^i$ are the Higgs branch operators that appear in the SC index (see table 3.2) for the i^{th} $T_{N,k}$ and Δ_{UV} corresponds to the dimension given in table 3.2. Finally, Φ_A are the adjoint chiral superfields belonging to the A^{th} $\mathcal{N} = 2$ vector multiplet.

The operator dimensions in the IR for this theory are those given in table 3.4.

It is easy enough to see that it is impossible to form any gauge-invariant operators that violate the unitarity bound $R \geq \frac{2}{3}$. To determine the dimensions of the conformal manifolds for these theories we use the method of Leigh and Strassler [23] (or equivalently [24]). From table 3.4 we see that there are $4g - 4$ marginal operators (where g is the genus of the quiver), all of the form $\mu\Phi$. Also, the number of gauge coupling constants is $3g - 3$. Finally, there are $4g - 5$ constraints coming from fixing anomalous dimensions: $2g - 2$ from the $T_{N,k}$'s plus $2g - 2$ from the adjoint chiral superfields minus one overall linear

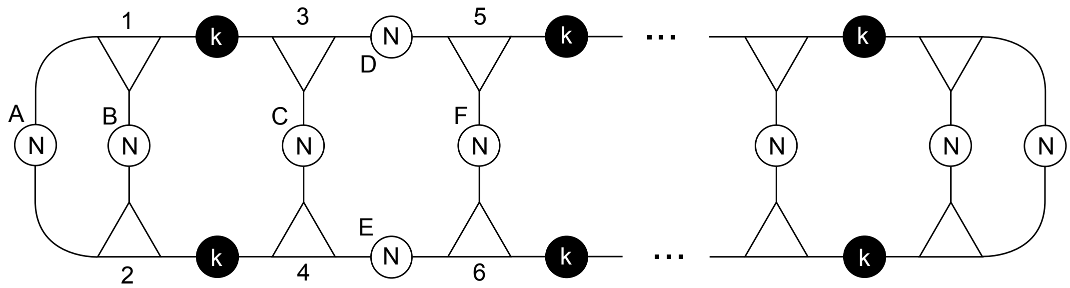

Figure 3.15: The generalised quiver for a theory with no $\mathcal{N} = 1$ $SU(N)$ vector multiplets.

combination. This means that the dimension of the conformal manifold for these theories is $3g - 2$. At present, we lack a geometric understanding for this number.

3.6.4 A Genus Three Example

We now look at a particular example where the conformal manifold exhibits a peculiar behaviour. The theory we wish to consider is given by the quiver diagram in figure 3.16.

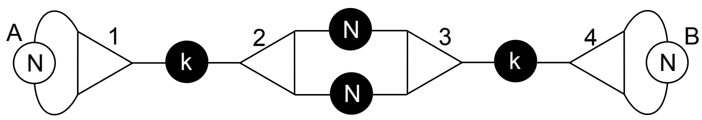

Figure 3.16: A generalised quiver for a theory with genus three. This theory has 4 $T_{N,k}$'s, 2 $\mathcal{N} = 2$ $SU(N)$ vector multiplets, 2 $\mathcal{N} = 1$ $SU(N)$ vector multiplets, and 2 $\mathcal{N} = 1$ $SU(k)$ vector multiplets.

Again, by using the rules in (3.38) and symmetry of the quiver we find that the IR R symmetry for this theory is

$$R_{IR} = R_{\mathcal{N}=1} + \left(\frac{k}{k+1} - \frac{5}{6}\right)(J_1 + J_4) + \frac{1}{6}(J_2 + J_3) + 2\left(\frac{k}{k+1} - \frac{5}{6}\right)(F_A + F_B)\,. \quad (3.42)$$

The operators in this theory in the IR and their R-charges are shown in table 3.5.

We once again use the method of Leigh and Strassler [23] to determine the dimension of the conformal manifold. We begin by counting the number of marginal operators. From table 3.5 we see that there are 6 operators, μ_2^2, μ_3^2 and $\mu_2\mu_3$ which are all marginal. These come from the 3 μ's associated to each $T_{N,k}$, related by chiral ring relations [7], each transforming in the adjoint of one part of the flavour symmetry. Furthermore, there are 4 marginal operators $\mu_1\Phi_A$, $\mu_4\Phi_B$. There are also 6 gauge coupling constants, which means that the total number of marginal parameters is 16. There are 5 constraints: 4 from fixing the anomalous dimensions of the T_N's, 2 from fixing the anomalous dimensions of

[7]This means that there are 3 $\mathrm{Tr}(\mu_1^2)$ operators and 3 $\mathrm{Tr}(\mu_2^2)$ operators related by the chiral ring relation: $\mathrm{Tr}(\mu_1^2) = \mathrm{Tr}(\mu_2^2)$ for each T_N, where $\mu_{1,2}$ transform in the adjoint of the $SU(N)$ factors of the flavour symmetry. See [4]. Also there are 2 $\mathrm{Tr}(\mu_2\mu_3)$ operators.

the adjoint chiral superfields, minus one overall linear combination. This means that the dimension of the conformal manifold is 11.

There is, however, a puzzle. When $k=3$, the number of marginal deformations increase, because the Tr $\left(\Phi_A^4\right)$ and Tr $\left(\Phi_B^4\right)$ operators and the $u_4^{1,4}$ operators (of which there are 4) become marginal. The number of constraints stays the same, and so the dimension of the conformal manifold increases from 11 to 17. This seems a bit strange, since from a geometrical point of view, there is no obvious reason why the $k=3$ theory should be any different from the theories with general k. Perhaps this is evidence that these theories are not good SCFTs, but without an AdS dual, it is difficult to say for sure.

Operator	R-charge
$\mu_{1,4}$	$3-2\frac{k}{k+1}$
$\mu_{2,3}$	1
$u_n^{(1,4)}$	$n\left(-1+2\frac{k}{k+1}\right)$
$u_n^{(2,3)}$	n
$\mathcal{O}_H^{1,4}$	$\Delta_{UV}\left(\frac{3}{2}-\frac{k}{k+1}\right)$
$\mathcal{O}_H^{2,3}$	$\Delta_{UV}\left(\frac{1}{2}\right)$
$\Phi_{A,B}^n$	$n\left(-1+2\frac{k}{k+1}\right)$

Table 3.5: The operators of the theory of section 3.6.4 along with their IR R-charges.

3.6.5 Another Subclass of Theories

Inspired by the results of the previous section, we now look at a family of theories that generalise those of the last section. Specifically, we look at theories whose quiver diagrams have the structure given in figure 3.17.

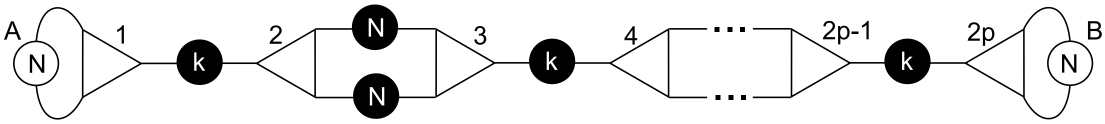

Figure 3.17: The generalised quiver diagram for a subclass of theories. There are $2p = 2g-2$ $T_{N,k}s$ where g is the genus of the quiver. Only the two end nodes represent $\mathcal{N}=2$ gauge groups; the rest are $\mathcal{N}=1$.

The R-symmetry for this theory is

$$R_0 = R_{\mathcal{N}=1} + \sum_{i=1}^{2p} \alpha_i J_i + \sum_{C=A,B} \beta_C F_C, \tag{3.43}$$

where

$$\alpha_{2n} = \frac{1}{6} - \frac{n-\frac{1}{2}p}{k+1} \qquad\qquad \alpha_{2n-1} = \frac{1}{6} + \frac{n-\frac{1}{2}p-1}{k+1}. \tag{3.44}$$

The dimensions of various operators in the theory are given in table 3.6. From this table is is easy to construct unitarity violating operators. For instance, we may construct one of these theories with $T_{N,k}$'s and increase the number of $T_{N,k}$'s until $p=k+1$. For this theory the $\mathrm{Tr}(\Phi_{A,B}^m)$ operators have dimension 0. Since the number of problematic operators increases in the large N limit, these theories are most likely not SCFTs.

It is interesting, though disappointing, that these theories (that is, all the theories considered in Section 3.6, not just the ones in this subsection), do not fall into a neat classification like their T_N counterparts do. As we have seen, some of the theories we attempted to build out of $T_{N,k}$'s do not appear to be good SCFTs. It would be interesting to find an organisational principle, such as the one in [30], that allows us to construct an obvious family of SCFTs. It is of course possible that no such theories are actually conformal, and perhaps the puzzle over the counting of marginal deformations discussed in the previous subsection is evidence of this. To conclusively solve this puzzle once and for all, we would need a method for constructing the AdS duals to these theories, and such an understanding is still lacking. Until then, we will have to regard the results of the present work as merely preliminary.

Operator	R-charge
μ_{2n}	$1+\frac{2n-p}{k+1}$
μ_{2n-1}	$1-\frac{2n-p-2}{k+1}$
$u_m^{(2n)}$	$m\left(1-\frac{2n-p}{k+1}\right)$
$u_m^{(2n-1)}$	$m\left(1+\frac{2n-p-2}{k+1}\right)$
$\mathcal{O}_H^{(2n)}$	$\Delta_{UV}\left(\frac{1}{2}-\frac{n-\frac{1}{2}p}{k+1}\right)$
$\mathcal{O}_H^{(2n-1)}$	$\Delta_{UV}\left(\frac{1}{2}-\frac{n-\frac{1}{2}p-1}{k+1}\right)$
$\Phi^m_{A,B}$	$m\left(1-\frac{p}{k+1}\right)$

Table 3.6: The operators of the theory in section 3.6.5 along with their conformal dimensions.

3.7 Conclusions

In this work, we investigated various properties of an interesting infinite family of theories of class $\mathcal{S}$, which we call the $T_{N,k}$ theories. These theories generalise Gaiotto's T_N theory and naturally arise when considering various S-duality frames of curves with two maximal punctures and multiple minimal punctures. Using techniques from duality as well as the superconformal index we described various properties of these theories, such as their global anomalies, central charges, and various operators and dimensions. We then used these theories as building blocks for constructing new $\mathcal{N}=1$ SCFTs, and checked whether the various theories under consideration appeared to be good conformal theories.

Our work raises some interesting questions. The most pressing is, of course, whether or not there exist AdS duals to these theories. Even in the case of the earlier work [34], it still remains unclear whether or not AdS duals for S_ℓ theories exist, and our understanding of the $\mathcal{N}=1$ AdS duals with punctured surfaces remains very incomplete. In order to establish beyond a reasonable doubt the existence of the SCFTs in this work as well as [34], it remains a pressing problem to find such duals. This would presumably also help us understand the dimension of the conformal manifold, a quantity for which at present we lack a geometric understanding for these theories.

Another interesting question is whether or not there exists a general principle, like the ones found for the B^{3}W theories, we could use for building analogous quivers out of the $T_{N,k}$'s. We were unable to find such a general principle, though it is possible that one exists. It is also possible that the absence of such a principle, as well as a seeming mismatch between various quantities of interest (*e.g.*, the dimension of the conformal manifold) in what would naively be considered different duality frames, may indicate that these theories are not indeed good SCFTs. On the other hand, it is also possible that there is an interesting geometric reason why the $\mathcal{N} = 1$ theories we consider here do not allow the full range of dualities found in the analogous $\mathcal{N} = 2$ cases, and in the absence of a good geometric understanding of these constructions, it may indeed be the most likely possibility that no such dualities exist. This possibility is especially tantalising, since understanding the geometric origin of such an obstruction would no doubt be of great interest.

The larger question explored by this work is which $\mathcal{N} = 1$ SCFTs can be built out of class $\mathcal{S}$ building blocks. As we know from our study of general $\mathcal{N} = 1$ theories with weakly coupled matter, it is no easy task to determine when a theory reaches a conformal fixed point in the IR. However, it is not outside the realm of possibility that, by using the techniques employed in this work as well as others, we could find large new tracts of the landscape of $\mathcal{N} = 1$ SCFTs.

Chapter 4

Chiral Ring Generating Functions & Moduli Space

This chapter is based on the paper [2].

4.1 Introduction

The study of D3-branes transverse to conical non-compact Calabi-Yau spaces has a long and storied past. As the transverse geometry has increased in complexity from $\mathbb{C}^3$ [57] to the conifold [58] to orbifold singularities [59, 60] and beyond [61, 62, 63], our understanding of the related world-volume theories, as well as the techniques used to study them, has increased dramatically. The motivations for these studies have ranged across a variety of themes: the fundamentals of D-brane physics, matrix models [64], brane-engineering of gauge theory dynamics, geometric engineering and reverse geometric engineering [65], and AdS/CFT [66, 67, 68]. Nevertheless, despite so many years of study, many interesting and important questions about these theories remain.

A particularly interesting part of any $\mathcal{N} = 1$ supersymmetric theory is the set of chiral gauge-invariant operators. These operators are annihilated by the supersymmetry generators of one chirality, $\bar{Q}$, and are usefully considered modulo an equivalence relation where commutators with $\bar{Q}$, i.e. $\bar{Q}$-exact operators, are set to zero. With this equivalence, derivatives can be set to zero. The ring formed by these operators, called the chiral ring, will play a pivotal role in the current work. In SCFTs, chiral primary operators (the lowest weight states in their representation of the conformal group) can be chosen as representatives of the chiral ring equivalence classes. We will focus on the chiral ring operators which are constructed from matter multiplets.

One particularly interesting question is how to derive the spectrum of chiral primary operators. For D3-branes at the tip of a Calabi-Yau cone, the dimensions of such operators can be computed with a-maximisation [20] (or, on the geometry side, Z-minimisation [69]). Additionally, the dimension Δ of two gauge-invariant chiral primary operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are additive in the sense that $\Delta(\mathcal{O}_1\mathcal{O}_2) = \Delta(\mathcal{O}_1) + \Delta(\mathcal{O}_2)$. A central question about such operators is then how many there are with a given dimension. This counting has been achieved in many theories thanks to the "plethystic program" of [70, 71]. In many cases of interest, the theories have a number of $U(1)$ global symmetries. A basis of the chiral ring can be formed from operators with definite charges, and generating functions can be defined for this refined counting.

A closely related object of interest in a supersymmetric gauge theory is the moduli space of constant (space-time independent) zero energy configurations of the scalar matter fields. Since the energy is a sum of squares of F- and D-terms, these configurations solve D- and F-term equations. Vacuum expectation values of gauge-invariant chiral ring operators can be used to parametrise the moduli space. As a result, the chiral ring is expected to be the ring of holomorphic polynomial functions on the moduli space (see e.g. [72, 73]). This connection between the space and the ring is of the form one encounters in algebraic geometry, where the study of ideals in the ring is an important part of the story.

It is natural to interpret the moduli space of the world-volume theory of a stack of D3-branes as the transverse geometry. For example, in the case of a single, flat D3-brane in $\mathbb{R}^{9,1}$, the three (uncharged) chiral superfields in the world-volume theory naturally correspond to the coordinates on the transverse space $\mathbb{C}^3$. For multiple branes, the moduli space is expected to be a symmetric product $Sym^N(X)$, where X is the transverse space and N is the number of branes; similarly, the gauge group of the world-volume gauge theory is a product of $U(N)$ factors. The ring of functions on the symmetric product corresponds to bosonic wavefunctions of an N-particle system on X. The explicit demonstration for $\mathbb{C}^3$ is in [57] and for the conifold in [58]; the moduli spaces for orbifold theories are also considered in [59, 74]. The appearance of symmetric products plays an important role in matrix theory [64] as well as reverse geometric engineering [65]. In the large N limit, the ring of functions on $Sym^N(X)$ can be mapped to a Fock space of states obtained by acting on a vacuum with oscillators, one for each holomorphic monomial function on the space X. The emergence of Fock spaces at large N is central to the AdS/CFT correspondence. On the AdS side, the Fock spaces arise from multi-particle states obtained from Kaluza-Klein reduction on the base of the cone transverse to the 3-branes. The counting of Fock space states is related to the counting of single particle states by the plethystic exponential and, as such, this has played an important role in the plethystic program [71, 70, 75]. The problem of counting chiral operators in M2-brane world-volume gauge theories transverse to orbifold geometries was also considered in [76]. A similar problem of calculating the superconformal index [28, 27] for D3-brane world-volume gauge theories has been completed for the transverse geometries $\mathbb{C}^3$ [28], the conifold [77], and other orbifold theories [78].

The goal of the present work is to explore these relationships between chiral rings, moduli spaces, large N Fock spaces, and the transverse geometry in various examples of D3-branes transverse to non-compact Calabi-Yau spaces. We will pay particular attention to the fact that, even in very simple cases such as $\mathbb{C}^3/\mathbb{Z}_2$, the existence of multiple branches of the moduli space brings additional subtleties to the web of inter-relations linking gauge theory combinatorics to geometry. Along the way, we will also examine the close relation observed [70] between the chiral ring of the $U(1)$ theory and the single trace operators in the large N theory (throughout this work, we will refer to the large N theory as the $U(\infty)$ theory). In this chapter we will only consider mesonic operators since the gauge group will always be a product of unitary gauge groups. The baryonic branch has been considered for similar theories with special unitary gauge groups in [79, 80].

The outline of the remainder of this chapter is as follows. In Section 2, we review some basic technology for quiver theories, chiral rings, and the generating functions that count chiral primaries. The remaining sections then consider a variety of examples in increasing order of complexity: $\mathcal{N} = 4$ (Section 3), the conifold (Section 4), $\mathbb{C}^3/\mathbb{Z}_2$ (Section 5), $\mathbb{C}^3/\mathbb{Z}_n$ (Section 6), and $\mathbb{C}^3/\hat{A}_n$ (Section 7), where $\hat{A}_n$ is the order n cyclic subgroup of $SU(2)$. Finally, in Section 8, we briefly conclude, and various details are relegated to appendices.

4.2 Review

Throughout this chapter we will be looking at gauge theories that live on the world-volume of a flat stack of D3-branes with a 3-complex dimensional transverse space. Our main goal in this work is to describe the relationships between the following objects:

- The space transverse to the D3-branes
- The moduli space of the world-volume theory
- The $U(1)$ (*i.e.*, single-brane) chiral ring of the world-volume theory
- The set of single-trace operators of the world-volume theory
- The ring of multi-trace operators of the world-volume theory
- The generating functions for counting operators in these sets/rings

In this section, we review these concepts.

4.2.1 Moduli Space

It is common for supersymmetric theories to have a moduli space of supersymmetric vacua. For a SUSY gauge theory with gauge group G, three different but equivalent ways of finding the classical moduli space are:

1. Solve the D- and F-term relations modulo G transformations.
2. Solve the F-term relations modulo $G_{\mathbb{C}}$ transformations, where $G_{\mathbb{C}}$ is the complexified gauge group.
3. Find the holomorphic gauge-invariant monomials modulo algebraic relations.

For a review, see *e.g.* [81, 73]. Throughout the present work, we will use methods 1 and 3. In the context of the moduli space, we will be talking about the ring of holomorphic polynomials on the space, so we now introduce some basic ideas in ring theory.

4.2.2 Rings, Ideals, and Quotient Rings

A ring $\mathcal{R}$ is a set of elements with two binary operations: addition and multiplication. The ring is an abelian group under addition $(\mathcal{R}, +)$ and a monoid under multiplication $(\mathcal{R}, \cdot)$, *i.e.* there is not necessarily a multiplicative inverse. Additionally, multiplication is distributive under addition. The rings we consider in this chapter will all be commutative under multiplication.

An ideal $\mathcal{I}$ is any subset of a ring which along with the addition operation $(\mathcal{I}, +)$ forms a subgroup of $(\mathcal{R}, +)$ and satisfies

$$\forall x \in \mathcal{I}, \forall y \in \mathcal{R} : x \cdot y \in \mathcal{I} \text{ and } y \cdot x \in \mathcal{I}. \tag{4.1}$$

The ideal generated by a set of elements $\{X_i\}$, is denoted $\langle X_i \rangle$ and is the minimal ideal containing the elements X_i; more precisely, $\langle X_i \rangle$ is the intersection of all ideals containing $\{X_i\}$. In other words the elements of an ideal generated by $\{X_i\}$ are $\sum_i a_i X_i$ for all possible $a_i \in \mathcal{R}$.

For any ring $\mathcal{R}$ and ideal $\mathcal{I}$, the quotient ring $\mathcal{R}/\mathcal{I}$ is the ring $\mathcal{R}$ modulo an equivalence relation which identifies two elements if their difference is an element of $\mathcal{I}$. As a simple example, consider $\mathbb{C}^2$ with coordinates (x, y). The space of holomorphic polynomials on $\mathbb{C}^2$ with complex coefficients corresponds to the ring of polynomials in two variables with complex coefficients, $\mathbb{C}[x, y]$[1]. The space of holomorphic polynomials on $\mathbb{C}^2$ has as a linear basis of monomials of the form $x^m y^n$, with $m, n \in \mathbb{Z}_{\geq 0}$. One ideal of $\mathbb{C}[x, y]$ is the ideal generated by y, $\mathcal{I} = \langle y \rangle$. This ideal contains y and anything with a factor of y in it. The quotient ring $\mathbb{C}[x, y]/\mathcal{I}$ has a linear basis monomials of the form x^m, with $m \in \mathbb{Z}_{\geq 0}$. In other words, $\mathbb{C}[x, y]/\langle y \rangle \cong \mathbb{C}[x]$.

4.2.3 Chiral Ring

We now review some basic facts about chiral rings. For reviews see [82, 83].

A chiral operator is any operator that is annihilated by the supersymmetry generators of one chirality, $\overline{Q}_{\dot{\alpha}}$. The OPE of chiral operators is non-singular and thus we can define a ring of chiral operators with a multiplication operation. Since an OPE of chiral operators does not depend on the positions of the operators, cluster decomposition implies that the OPE only depends on the vevs of fields. Thus, within the chiral ring, operators with the same vev are considered equivalent. As a consequence of the vacuum being

[1] Throughout this chapter when we refer to the ring of holomorphic polynomials on a space we mean the space of holomorphic polynomials on this space with multiplication and addition defined in the usual way. Also, we will discuss the generating function for the ring of holomorphic polynomials on a space. This generating function will have one term for each basis holomorphic monomial in the ring.

annihilated by supersymmetry generators, chiral operators should be considered equivalent if they differ by a term of the form $\{\overline{Q}_{\dot{\alpha}}, \dots]$; two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are equivalent if $\mathcal{O}_1 = \mathcal{O}_2 + \{\overline{Q}_{\dot{\alpha}}, X^{\dot{\alpha}}]$.

In superspace, the condition that a superfield Φ is a chiral superfield is $\overline{D}_{\dot{\alpha}}\Phi = 0$. Two chiral operators being equivalent if they differ by $\{\overline{Q}_{\dot{\alpha}}, ...]$ implies that the two chiral superfields X_1, X_2 they belong to are equivalent if $X_1 = X_2 + \overline{D}_{\dot{\alpha}}\overline{D}^{\dot{\alpha}}Z$. In Wess-Zumino models the equation of motion of a chiral superfield Φ is

$$\partial_\Phi W(\Phi) = \overline{D}_{\dot{\alpha}}\overline{D}^{\dot{\alpha}}\overline{\Phi}, \tag{4.2}$$

from which we can see that in the chiral ring the F-term relations

$$\partial_\Phi W(\Phi) = 0 \tag{4.3}$$

are satisfied. If we take the gauge-variant F-terms and contract with all possible operators that result in gauge-invariant terms then we can define the ideal, $\mathcal{I}_0$, which is generated by these gauge-invariant terms. Then if $\mathcal{R}_0$ is the ring of chiral gauge-invariant operators, $\mathcal{R} = \mathcal{R}_0/\mathcal{I}_0$ is the chiral ring of a theory with non-zero superpotential. For a theory with no superpotential there are no F-terms and so the chiral ring is $\mathcal{R} = \mathcal{R}_0$.

For the theories we study in this chapter, all elements of the chiral ring will be either single- or multi-trace operators, *i.e.,* a single trace of products of operators or several single-trace operators multiplied by each other. We will not consider determinants because the theories in questions will have unitary, not special unitary, gauge groups.

4.2.4 Generating Functions and Plethystics

In [70, 71] the authors describe a method of counting operators in a theory using generating functions. Such a generating function typically looks like

$$f(t_1, \dots, t_k) = \sum_{i_1,\dots,i_k} c_{i_1,\dots,i_k} t_1^{i_1} \dots t_k^{i_k}, \tag{4.4}$$

where t_i is the fugacity (chemical potential) for the i-th quantum number and $c_{i_1,\dots i_k}$ gives the number of operators with quantum numbers $(i_1, \dots, i_k)$.

A useful tool in [70] is the "plethystic exponential", which is used to get the generating function for multi-trace operators from the generating function for single-trace operators at large N. If we have some function $F_S(t_i)$, the plethystic exponential of the function is defined to be

$$\text{P.E.}\left[F_S(t_i)\right] = \exp\left\{\sum_{k=1}^{\infty} \frac{F_S(t_i^k) - F_S(0)}{k}\right\}. \tag{4.5}$$

To see how this gives the multi-trace operator generating function from the single-trace operator generating function, consider a generating function for single-trace operators where each single-trace operator has a different chemical potential. In this case, $F_S(t_i) =$

$\sum_i t_i$. The plethystic exponential is thus

$$F_M(t_i) = \text{P.E.}\left[F_S(t_i)\right] = \prod_i \frac{1}{1-t_i}. \tag{4.6}$$

This will give one term for each way the t_i can be raised to different powers and multiplied, so this is indeed the generating function for multi-trace operators.

In [70], the authors identify the set of large N single-trace operators with the set of holomorphic polynomials on the moduli space, which in turn is identified with the set of holomorphic polynomials on the transverse space. They use this logic to derive the generating function for single-trace operators using results from algebraic geometry. We will find that these relations between moduli space, transverse geometry and single traces are only true modulo subtleties due to the existence of multiple branches of moduli space which we will describe.

4.3 $\mathcal{N} = 4$ SYM

We begin with $U(N)$ $\mathcal{N} = 4$ SUSY Yang-Mills (SYM) as a particularly simple example which will illustrate the ideas used throughout the remainder of the chapter. This is the world-volume theory on a flat stack of N D3-branes, with transverse space $\mathbb{C}^3$. In $\mathcal{N} = 1$

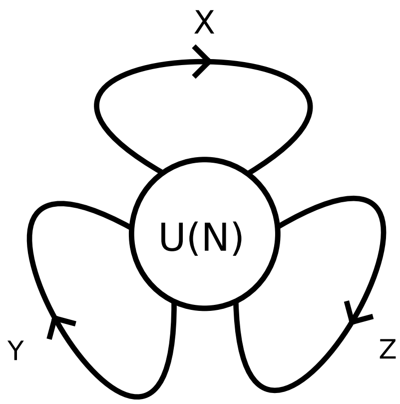

Figure 4.1: The $\mathcal{N} = 1$ quiver diagram for $\mathcal{N} = 4$ SYM.

language, we can write the theory as a $U(N)$ gauge theory with three adjoint chiral superfields X, Y, Z whose quiver diagram is given in figure 4.1. The $\mathcal{N} = 1$ superpotential is

$$W = \text{Tr}\left(X\left[Y, Z\right]\right), \tag{4.7}$$

which yields the F-term equations

$$XY = YX, \qquad XZ = ZX, \qquad YZ = ZY, \tag{4.8}$$

where we have suppressed gauge indices.

4.3.1 $N = 1$ Moduli Space

The F-terms enforce that the matrices X, Y and Z all commute. This means that they can be simultaneously put in to upper triangular form by a unitary transformation. The D-term constraint

$$[X, X^\dagger] + [Y, Y^\dagger] + [Z, Z^\dagger] = 0 \tag{4.9}$$

then enforces that X, Y and Z must be diagonal. After diagonalising, there is still a residual S_N gauge symmetry which interchanges the eigenvalues, so the moduli space is $(\mathbb{C}^3)^N/S_N$ or $\mathrm{Sym}^N(\mathbb{C}^3)$.

For a single brane, $N = 1$, so the moduli space is just $\mathbb{C}^3$. This demonstrates the first relationship we would like to highlight: the $N = 1$ moduli space of the D3-brane world-volume theory is the transverse space. For multiple branes, one can interpret the S_N action as swapping the positions of the N D3-branes in the transverse space $\mathbb{C}^3$, with the same result.

4.3.2 $W = 0$ Large N Chiral Ring

It is interesting to consider this theory when the superpotential is turned off but the gauge coupling remains non-zero; this breaks $\mathcal{N} = 4$ SUSY but preserves $\mathcal{N} = 1$. In this situation, we now look to find the generating function for multi-trace operators. Now, the operators X, Y, Z do not commute. Thus, the single-trace operators in the theory consist of various configurations of the adjoint chiral superfields with given orderings. The generating function for single-trace-operators in the large N theory is then given by the generating function for 3-ary necklaces of beads[2], a problem whose solution can be found in combinatorics. The generating function is found using the Pólya enumeration theorem [84, 85][3], but before describing this theorem we first introduce a few ideas.

For a finite group $G \subseteq S_n$ the cycle index is defined as

$$Z_G(t_1, t_2, ..., t_n) \equiv \frac{1}{|G|} \sum_{g \in G} t_1^{j_1(g)} t_2^{j_2(g)} ... t_n^{j_n(g)}, \tag{4.10}$$

where $j_i(g)$ is the number of cycles of length i in g. Let X be a set of n objects and let G be a finite group that acts on X. Additionally, let $Y = \{c_1, ..., c_k\}$ be a set of $|Y| = k$ colours so that Y^X is the set of coloured arrangements of these n objects. The colour generating function is defined to be $f(c_1, ..., c_k) \equiv \sum_{i=1}^k c_i$. Then the Pólya enumeration theorem counts the number of orbits under G of the coloured arrangements of n beads.

[2] A necklace is an arrangement of objects (or beads) that is invariant under the action of the cyclic group. The generating function for k-ary necklaces of beads counts how many necklaces we can construct using beads of k different colours.

[3] An introductory treatment of the theorem along with a wide range of applications can be found in [86].

According to the theorem, the counting is given by the generating function

$$F_G\left(c_1,...,c_k\right) = Z_G\left(f(c_i), f(c_i^2), ..., f(c_i^n)\right). \tag{4.11}$$

We are interested in counting k-ary necklaces of n beads so for this case the finite group is the cyclic group $G = C_n$. The cyclic group C_n has $\varphi(d)$ elements of order d for each divisor d of n, where $\varphi(d)$ is the Euler totient function[4]. Thus the cycle index is

$$Z_{C_n}\left(t_1,...,t_n\right) = \frac{1}{n}\sum_{d|n}\varphi(d)\left(t_d\right)^{\frac{n}{d}}. \tag{4.12}$$

This means that the generating function for k-ary necklaces of n beads is given by

$$F_{C_n}\left(c_1,...,c_k\right) = \frac{1}{n}\sum_{d|n}\varphi(d)\left(\Sigma_{i=1}^k c_i^d\right)^{\frac{n}{d}}, \tag{4.13}$$

and the generating function for k-ary necklaces of any number of beads is

$$F_C\left(c_1,...,c_k\right) = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{d|n}\varphi(d)\left(\Sigma_{i=1}^k c_i^d\right)^{\frac{n}{d}} = -\sum_{d=1}^{\infty}\frac{\varphi(d)}{d}\log\left[1-\left(\Sigma_{i=1}^k c_i^d\right)\right]. \tag{4.14}$$

For the case of $\mathcal{N} = 4$ SYM with $W = 0$, the generating function for single-trace operators is exactly the generating function for 3-ary necklaces of beads (*i.e.*, set $k = 3$ in the previous formula):

$$\boxed{F_S^{(\infty)}\left(x,y,z\right) = -\sum_{d=1}^{\infty}\frac{\varphi(d)}{d}\log\left[1-\left(x^d+y^d+z^d\right)\right]}. \tag{4.15}$$

We can then get the generating function for multi-trace operators by taking the plethystic exponential of this function

$$F_M^{(\infty)}\left(x,y,z\right) = \text{P.E.}\left[F_S^{(\infty)}\left(x,y,z\right)\right] = \exp\left\{\sum_{k=1}^{\infty}\frac{1}{k}F_S^{(\infty)}\left(x^k,y^k,z^k\right)\right\}, \tag{4.16}$$

so that we get

$$\boxed{F_M^{(\infty)}\left(x,y,z\right) = \prod_{n=1}^{\infty}\frac{1}{1-\left(x^n+y^n+z^n\right)}}. \tag{4.17}$$

These results were originally found in [87].

We can get back to the single-trace operator generating function by using the plethystic

[4]The Euler totient function $\varphi(n)$ counts the number of positive integers less than or equal to n that are co-prime to n.

logarithm:

$$F_S^{(\infty)}(x,y,z) = PE^{-1}\left[F_M^{(\infty)}(x,y,z)\right] = \sum_{k=1}^{\infty}\frac{\mu(k)}{k}\log\left(F_M^{(\infty)}(x^k,y^k,z^k)\right) \tag{4.18}$$

and using the identity

$$\frac{\varphi(n)}{n} = \sum_{d|n}\frac{\mu(d)}{d}. \tag{4.19}$$

4.3.3 $W \neq 0$ Large N Chiral Ring

We now look to find the generating function for multi-trace operators in the theory with non-zero superpotential. Turning on the superpotential enforces the commutativity of the adjoint chiral superfields. The generating function for single-trace operators with zero superpotential in equation (4.15) is of the form

$$F_S^{(\infty)}(x,y,z) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{n_1,n_2,n_3}x^{n_1}y^{n_2}z^{n_3}, \tag{4.20}$$

which counts the number c_{n_1,n_2,n_3} of single-trace operators that we can make with n_1 X's, n_2 Y's and n_3 Z's. When we enforce the commutativity of operators, all these coefficients are 1. Thus the single-trace operator generating function for large N $\mathcal{N}=4$ SYM with non-zero superpotential is

$$\boxed{F_S^{(\infty)}(x,y,z) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} x^{n_1}y^{n_2}z^{n_3} = \frac{1}{1-x}\frac{1}{1-y}\frac{1}{1-z}}. \tag{4.21}$$

This generating function is equal to the generating function for the ring of holomorphic polynomials on $\mathbb{C}^3$. We also could have obtained this formula using the Pólya enumeration theorem with $G = S_n$; see Appendix A.3.

We can once again take the plethystic exponential of equation (4.21) to give us the generating function for multi-trace operators in large N $\mathcal{N}=4$ SYM with non-zero superpotential:

$$\boxed{F_M^{(\infty)}(x,y,z) = \prod_{n=1}^{\infty}\prod_{n_1=0}^{n}\prod_{n_2=0}^{n-n_1}\frac{1}{1-x^{n_1}y^{n_2}z^{n-n_1-n_2}}}. \tag{4.22}$$

This formula could have alternately been derived from first principles by using $F_M^{(\infty)} = \prod_i (1-t_i)^{-1}$, where the product is over all single-trace operators.

$U(\infty)$ Fock Space

In this example, the generating function for the chiral ring of the $U(1)$ theory tells us the operator content of the theory; for $\mathcal{N}=4$ SYM this generating function is the generating function for holomorphic polynomials on $\mathbb{C}^3$. This generating function is also the generating function for the Hilbert space of a single boson on $\mathbb{C}^3$ and gives a basis of wavefunctions for a particle on $\mathbb{C}^3$. In the spirit of [64], we can interpret this Hilbert space in terms of wavefunctions of a single brane moving on the transverse space.

Equation (4.22) tells us that the generating function for the large N chiral ring is equal to the generating function for the multi-particle Fock space for bosons on $\mathbb{C}^3$ which is the Fock space of the multiple branes moving on the transverse space. More explicitly, the space of wavefunctions for the i-th boson on $\mathbb{C}^3$ is spanned by

$$\psi_{i,p,q,r}(x,y,z) = x^p_{(i)} y^q_{(i)} z^r_{(i)}. \tag{4.23}$$

The space of wavefunctions for two bosons is spanned by the symmetric sum

$$\frac{1}{2}\left(x^{p_1}_{(1)} y^{q_1}_{(1)} z^{r_1}_{(1)} x^{p_2}_{(2)} y^{q_2}_{(2)} z^{r_2}_{(2)} + x^{p_1}_{(2)} y^{q_1}_{(2)} z^{r_1}_{(2)} x^{p_2}_{(1)} y^{q_2}_{(1)} z^{r_2}_{(1)}\right). \tag{4.24}$$

This space has a one-to-one correspondence with the Hilbert space of two bosons on $\mathbb{C}^3$, which is spanned by

$$B^\dagger_{p_1,q_1,r_1} B^\dagger_{p_2,q_2,r_2}|0\rangle, \tag{4.25}$$

where $B^\dagger_{p,q,r}$ is the creation operator for a particle with wavefunction $x^p y^q z^r$, satisfying $[B^\dagger_{p_1,q_1,r_1}, B^\dagger_{p_2,q_2,r_2}] = 0$.

More generally, for n bosons the symmetrised wavefunctions

$$\frac{1}{n!}\left(\sum_{\sigma\in S_n}\prod_{i=1}^{n} x^{p_i}_{\sigma(i)} y^{q_i}_{\sigma(i)} z^{r_i}_{\sigma(i)}\right) \tag{4.26}$$

are in one-to-one correspondence with the Hilbert space of n bosons on $\mathbb{C}^3$ spanned by

$$B^\dagger_{p_1,q_1,r_1}\cdots B^\dagger_{p_n,q_n,r_n}|0\rangle. \tag{4.27}$$

By inspecting equation (4.22) one can see that

$$F^{(\infty)}_M(x,y,z) = F_{\mathrm{Fock}}(\mathbb{C}^3). \tag{4.28}$$

where F_{Fock} is the generating function for the Fock space of bosons on $\mathbb{C}^3$.

4.3.4 $N = 1$ Chiral Ring

The generating function for operators in the chiral ring of the $U(1)$ $\mathcal{N} = 4$ SYM theory is simply equal to the large N single-trace operator generating function given in equation (4.21). This is because there is a mapping which maps every operator in the $U(1)$ $\mathcal{N} = 4$ SYM theory to a single-trace operator in the $U(\infty)$ theory[5]. This mapping is

$$X^{n_1}Y^{n_2}Z^{n_3} \to \mathrm{Tr}(X^{n_1}Y^{n_2}Z^{n_3}). \tag{4.29}$$

[5]We use the notation $U(\infty)$ to denote the large N theory.

Although this mapping is rather intuitive here, we will see in later sections that naïve intuition fails for more complicated theories, and in fact the generating function for the chiral ring of the $U(1)$ theory is not equal to the large N single-trace operator generating function.

This highlights two more relationships that are part of this story. The first is that the $U(1)$ chiral ring is equal to the ring of holomorphic polynomials on the transverse space. The second is that the set of elements in the $U(1)$ chiral ring is equal to the set of single-trace operators in the large N gauge theory.

4.3.5 Conclusion

In this section, we observed the following relationships for $U(N)$ $\mathcal{N} = 4$ SYM:

1. The $N = 1$ moduli space is the same as the space transverse to the D3-branes.
2. The $U(1)$ chiral ring is equal to the ring of holomorphic polynomials on the moduli space, and thus equal to the ring of holomorphic polynomials on the transverse space.
3. The set of elements in the $U(1)$ chiral ring is equal to the set of single-trace operators in the $U(\infty)$ theory.
4. The multi-trace operator generating function for the $U(\infty)$ theory gives us a generating function for bosons moving on the transverse space.

In the coming sections we see while some of these relationships persist in more complicated examples, some of them do not.

4.4 Conifold

Our next example is a stack of N D3-branes transverse to the conifold $\mathcal{C}$, as studied in [58]. The world-volume theory is an $\mathcal{N} = 1$ $U(N) \times U(N)$ gauge theory with two chiral superfields, A_1 and A_2, in the $(\mathbf{N}, \overline{\mathbf{N}})$ representation and two chiral superfields, B_1 and B_2, in the $(\overline{\mathbf{N}}, \mathbf{N})$ representation. The quiver for this theory is given in figure 4.2.

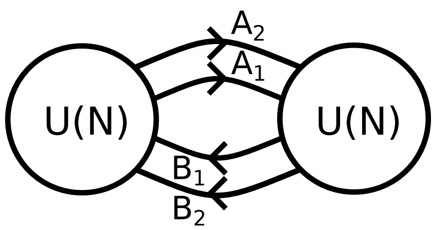

Figure 4.2: $\mathcal{N} = 1$ quiver diagram for the conifold theory.

The theory also has the superpotential

$$W = \mathrm{Tr}\,(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)\,, \tag{4.30}$$

so the F-term relations are

$$\begin{aligned} B_1A_2B_2 - B_2A_2B_1 &= 0, & B_1A_1B_2 - B_2A_1B_1 &= 0, \\ A_1B_2A_2 - A_2B_2A_1 &= 0, & A_1B_1A_2 - A_2B_1A_1 &= 0. \end{aligned} \tag{4.31}$$

Gauge-invariant operators must consist of combinations of

$$\begin{aligned} W &= A_1B_1, & X &= A_1B_2, \\ Y &= A_2B_1, & Z &= A_2B_2, \end{aligned} \tag{4.32}$$

which are in the $\mathbf{N} \otimes \overline{\mathbf{N}}$ representation of the one of the $U(N)$ gauge groups, although we have suppressed the indices.

The F-term relations expressed in terms of these are

$$[W, X] = [W, Y] = [W, Z] = [X, Y] = [X, Z] = [Y, Z] = 0 \tag{4.33}$$

and

$$WZ = XY. \tag{4.34}$$

4.4.1 $N = 1$ Moduli Space

In the $N = 1$ (*i.e.*, $U(1)^2$) theory, the superpotential vanishes. Thus we need only solve the D-term equation

$$D = |A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0, \tag{4.35}$$

which is the equation for the conifold. (We do not consider an FI term.) We once again see that the $N = 1$ moduli space is exactly the same as the space transverse to the brane.

4.4.2 $W = 0$ Large N Chiral Ring

For the theory with zero superpotential, the generating function for single-trace operators is given by the generating function for 4-ary necklaces:

$$\boxed{F_S^{(\infty)}(w, x, y, z) = -\sum_{d=1}^{\infty} \frac{\varphi(d)}{d} \log\left[1 - \left(w^d + x^d + y^d + z^d\right)\right]}. \tag{4.36}$$

Rewriting w, x, y, and z in terms of $a_{1,2}$ and $b_{1,2}$ via the relations in (4.32), the generating function is of the form

$$F_S^{(\infty)}(a_1, a_2, b_1, b_2) = \sum_{n=0}^{\infty} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} c_{n,n_1,n_2}, a_1^{n_1} b_1^{n_2} a_2^{n-n_1} b_2^{n-n_2}, \tag{4.37}$$

where c_{n,n_1,n_2} counts the number of single-trace operators that can be constructed using n_1 A_1 operators, $n-n_1$ A_2 operators, n_2 B_1 operators, $n-n_2$ B_2 operators. The number of A operators and the number of B operators must be equal for the single-trace operator to be gauge-invariant. The generating function for multi-trace operators is obtained by taking the plethystic exponential of (4.36):

$$\boxed{F_M^{(\infty)}(x,y,z) = \prod_{n=1}^{\infty} \frac{1}{1-(w^n+x^n+y^n+z^n)}}. \tag{4.38}$$

This is in agreement with [88].

4.4.3 $W \neq 0$ Large N Chiral Ring

All single-trace operators in the conifold theory have alternating A's and B's, *i.e.* they are of the form $\mathrm{Tr}(ABAB...AB)$. Turning on the superpotential means that the F-term relations allow us to organise the trace so that the first n_1 A operators are A_1's and the last $n-n_1$ A operators are A_2's, and similarly for the B's. Thus the generating function is just the function in equation (4.37) with all the coefficients set to 1:

$$\boxed{F_S^{(\infty)}(a_1,a_2,b_1,b_2) = \sum_{n=0}^{\infty}\sum_{n_1=0}^{n}\sum_{n_2=0}^{n} a_1^{n_1} b_1^{n_2} a_2^{n-n_1} b_2^{n-n_2}}. \tag{4.39}$$

This formula has a closed form expression in terms of w, x, y and z:

$$F_S^{(\infty)}(w,x,y,z) = \frac{1}{w-x-y+z}\left(\frac{1}{1-w} - \frac{1}{1-x} - \frac{1}{1-y} + \frac{1}{1-z}\right). \tag{4.40}$$

The generating function for multi-trace operators can once again be found by taking the plethystic exponential of equation (4.39):

$$\boxed{F_M^{(\infty)}(a_1,a_2,b_1,b_2) = \prod_{n=0}^{\infty}\prod_{n_1=0}^{n}\prod_{n_2=0}^{n} \frac{1}{1-a_1^{n_1} b_1^{n_2} a_2^{n-n_1} b_2^{n-n_2}}}, \tag{4.41}$$

which can again be seen intuitively from $F_M^{(\infty)} = \prod_\Phi (1-\Phi)^{-1}$, where the product is over single-trace operators. Also as before, $F_S^{(\infty)}$ can alternately be derived from the Pólya enumeration theorem by taking the product of two generating functions for 2-ary necklaces.

If we take equation (4.40) and make the substitutions $w \to qa$, $x \to qb$, $y \to \frac{q}{b}$, and $z \to \frac{q}{a}$ we regain the form of the generating function presented in [70]:

$$F_S^{(\infty)}(a,b,q) = \frac{ab(q-1)(q+1)}{(a-q)(aq-1)(q-b)(bq-1)}. \tag{4.42}$$

This substitution is indicative of the relationships between the charges that we have chosen here and the charges that are chosen in [70].

$U(\infty)$ Fock Space

Equation (4.41) tells us that the generating function for multi-trace operators in the $U(\infty)$ theory is equal to the generating function for the Fock space of bosons on the conifold:

$$F_M^{(\infty)}(a_1, a_2, b_1, b_2) = F_{\text{Fock}}^{(\infty)}(\mathcal{C}). \tag{4.43}$$

As was the case for $\mathcal{N} = 4$ SYM, we can again interpret this as the Fock space for branes on the transverse space.

4.4.4 $N = 1$ Chiral Ring

As with $\mathcal{N} = 4$, there is a one-to-one mapping between operators in the $U(1)$ theory and single-trace operators in the $U(\infty)$ theory. This mapping is

$$A_1^{n_1} A_2^{n-n_1} B_1^{n_2} B_2^{n-n_2} \to \text{Tr}(A_1^{n_1} A_2^{n-n_1} B_1^{n_2} B_2^{n-n_2}). \tag{4.44}$$

Thus the generating function for the chiral ring of the $U(1)$ theory is equal to the generating function for single-trace operators in the $U(\infty)$ theory given in equation (4.39).

4.4.5 Conclusion

We see from this slightly more complicated example many of the same phenomena that we saw with $\mathcal{N} = 4$ SYM. First, the moduli space is equal to the transverse space, and the chiral ring of the $U(1)$ theory is the ring of holomorphic polynomials on this space. The chiral ring has the same elements as the set of single-trace operators in the large N theory. Also, in the $U(\infty)$ theory we can identify a Fock space of bosons which we interpret as the Fock space of the branes moving on the transverse space. In the next section, we will see that some of these relationships do not hold more generally.

4.5 $\mathbb{C}^3/\mathbb{Z}_2$

We now consider the theory living on the world-volume of N D3-branes probing a $\mathbb{C}^3/\mathbb{Z}_2$ singularity [74]. This theory has a quiver as in figure 4.3 and a superpotential

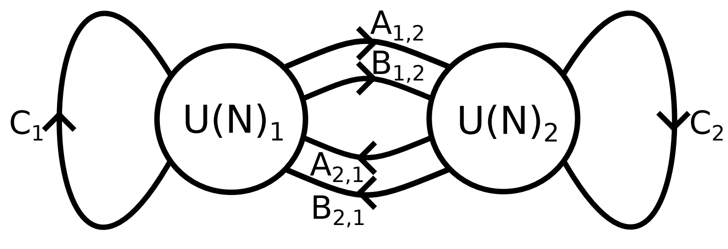

Figure 4.3: $\mathcal{N} = 1$ quiver diagram for the $\mathbb{C}_3/\mathbb{Z}_2$ theory.

given by

$$W = \mathrm{Tr}\left[C_1(A_{1,2}B_{2,1} - B_{1,2}A_{2,1}) + C_2(A_{2,1}B_{1,2} - B_{2,1}A_{1,2})\right]. \qquad (4.45)$$

From the quiver and superpotential we can see that this theory is in fact an $\mathcal{N} = 2$ theory, and can flow to the conifold theory via mass terms for the adjoint chiral superfields. The F-term relations are

$$\begin{aligned} C_1A_{1,2} &= A_{1,2}C_2, & C_1B_{1,2} &= B_{1,2}C_2, \\ C_2A_{2,1} &= A_{2,1}C_1, & C_2B_{2,1} &= B_{2,1}C_1, \\ A_{1,2}B_{2,1} &= B_{1,2}A_{2,1}, & A_{2,1}B_{1,2} &= B_{2,1}A_{1,2}. \end{aligned} \qquad (4.46)$$

It will prove useful to use the composite operators $W = B_{1,2}A_{2,1}$, $X = A_{1,2}A_{2,1}$, $Y = B_{1,2}B_{2,1}$ and $Z = A_{1,2}B_{2,1}$ throughout the remainder of this section.

4.5.1 $N = 1$ Moduli Space

The F-terms equations have two branches of solutions:

1. $\{X, Y, Z, C_1, C_2 \,|\, XY = Z^2, C_1 = C_2\}$,
2. $\{X, Y, Z, C_1, C_2 \,|\, X = Y = Z = 0\}$.

On the first branch the moduli space is described by the gauge-invariant operators X, Y, Z and C $(= C_1 = C_2)$ subject to $XY = Z^2$; this is just the space $\mathbb{C}^3/\mathbb{Z}_2$. On the second branch, C_1 does not necessarily equal C_2, and $X = Y = Z = 0$; this is the simply $\mathbb{C}^2$. The two branches intersect along the line $C_1 = C_2$ when $X = Y = Z = 0$. For a cartoon of the full moduli space, see figure 4.4.

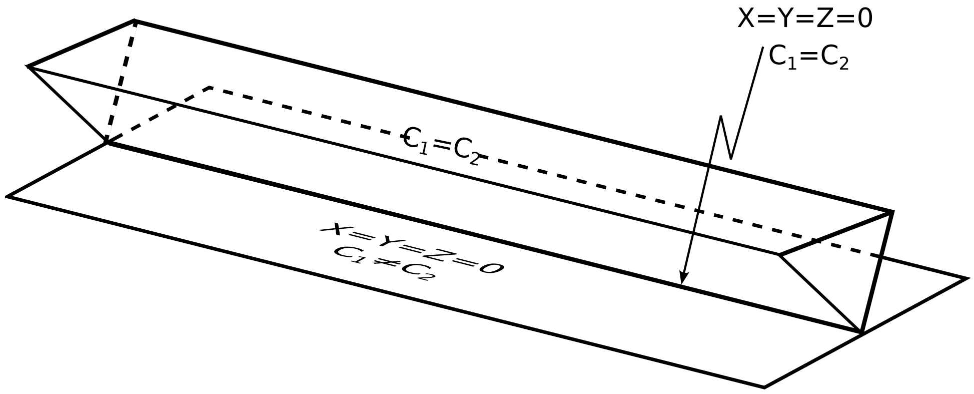

Figure 4.4: Moduli space of the $\mathbb{C}^3/\mathbb{Z}_2$ theory. The $C_1 = C_2$ branch is 3 complex dimensional and the $C_1 \neq C_2$ branch is 2 complex dimensional.

We denote the full moduli space $\mathbb{C}^3/\mathbb{Z}_2 \cup \mathbb{C}^2$, where the particular union that is meant is the one where the two spaces share the line $X = Y = Z = 0$, $C_1 = C_2$. In other words

$$\mathcal{M} = \mathbb{C}^3/\mathbb{Z}_2 \cup \mathbb{C}^2, \qquad \mathbb{C}^3/\mathbb{Z}_2 \cap \mathbb{C}^2 = \mathbb{C}. \tag{4.47}$$

In contrast to the previous two examples, we see here that the moduli space is not simply the space transverse to the D3-branes. There is one *main branch* which is the transverse space, but we also see the existence of an *extra branch* which has a different dimension than the main branch. This extra branch of moduli space is something we will see in later examples and has been observed, e.g. in [89, 75].

4.5.2 $W = 0$ Large N Chiral Ring

The $W = 0$ large N single-trace operator generating function can be found using the Pólya enumeration theorem, as in previous sections but this time counting 6-ary necklaces. One wrinkle is that because the fields C_1 and C_2 are not charged under the same gauge group, the fields should not be placed next to each other in a trace. In combinatorics language, we cannot place the beads of colour c_1 and c_2 next to each other in a necklace. However, this problem is easily solved by using the colour generating function

$$f(w, x, y, z, c_1, c_2) = w + x + y + z + c_1 + c_2 - c_1c_2, \tag{4.48}$$

where the final term subtracts the contribution from necklaces with adjacent c_1 and c_2 beads (w, x, y, z are as in the previous section). This yields

$$\boxed{F_S^{(\infty)}(w, x, y, z, c_1, c_2) = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log\left[1 - \left(w^k + x^k + y^k + z^k + c_1^k + c_2^k - c_1^kc_2^k\right)\right]}, \tag{4.49}$$

and taking the plethystic exponential gives us the multi-trace operator generating function

$$\boxed{F_M^{(\infty)}(w, x, y, z, c_1, c_2) = \prod_{k=1}^{\infty} \frac{1}{1 - \left(w^k + x^k + y^k + z^k + c_1^k + c_2^k - c_1^kc_2^k\right)}}. \tag{4.50}$$

This matches the formula given in [88].

4.5.3 $W \neq 0$ Large N Chiral Ring

We now turn on the superpotential in equation (4.45). The F-term equivalences preserve the number of A's, the number of B's, and the number of (C_1's $+C_2$'s), so our single-trace operator generating function can have at most three chemical potentials. All operators can be arranged using the F-term relations so that they have the form $\mathrm{Tr}\,(C_1^n)$, $\mathrm{Tr}\,(C_2^n)$ or $\mathrm{Tr}\,(C_1^m A_{1,2}A_{2,1}A_{1,2}B_{2,1}B_{1,2}...A_{1,2}B_{2,1})$, with the order of A's and $B's$ irrelevant. The generating function is then

$$\boxed{F_S^{(\infty)}(a,b,c)=\sum_{m=0}^{\infty}c^m\sum_{\ell=0}^{\infty}\sum_{k=0}^{2\ell}a^{2\ell-k}b^k+\sum_{m=1}^{\infty}c^m=\frac{1+ab}{(1-c)(1-a^2)(1-b^2)}+\frac{c}{1-c}}, \tag{4.51}$$

with plethystic exponential

$$\boxed{F_M^{(\infty)}(a,b,c)=\prod_{n=0}^{\infty}\prod_{\ell=0}^{\infty}\prod_{k=0}^{2\ell}\frac{1}{1-c^m a^{2\ell-k}b^k}\prod_{n=1}^{\infty}\frac{1}{1-c^n}}. \tag{4.52}$$

Using the methods in [70], one can derive the single-trace operator generating function

$$F_S^{(\infty)}(t)=\frac{1+t^2}{(1-t)^3(1+t)^2}. \tag{4.53}$$

We have included this short calculation in appendix A.4. To compare the result with our answer, we make the substitution $a\to t$, $b\to t$, $c\to t$, which yields

$$F_S^{(\infty)}(t)=\frac{t^5-2t^3+t^2+t+1}{(1-t)^3(1+t)^2}, \tag{4.54}$$

which is different than the earlier result.

It is straightforward to find the source of the discrepancy. In [70] it was assumed that the set of single-trace operators was equal to the set of elements in the ring of holomorphic polynomials on the moduli space and thus equal to the set of elements in the ring of holomorphic polynomials on the transverse space. As we have seen for the $\mathbb{C}^3/\mathbb{Z}_2$ theory, this is not quite correct. Instead we saw that the moduli space has two separate branches, only one of which is the space transverse to the D3-branes. Indeed, we find that subtracting the contribution from the $C_1\neq C_2$ branch reproduces the previous result.

We can alternatively see the difference in the two approaches from a ring theoretic perspective. For the full world-volume theory, the set of single-trace operators is not equal to the set of elements in the ring of holomorphic polynomials on $\mathbb{C}^3/\mathbb{Z}_2$, since $\mathrm{Tr}(C_1^k)$ and $\mathrm{Tr}(C_2^k)$ are not necessarily equal. Denoting the ring of gauge-invariant operators in the $W=0$ $\mathbb{C}^3/\mathbb{Z}_2$ theory by R_0, we consider the ring $R_W=\{R_0|\text{F-term equivalences}\}$, while the earlier work considered only $R'_W=\{R_W|C_1=C_2\}$, which is the ring of holomorphic polynomials on $\mathbb{C}^3/\mathbb{Z}_2$.

$U(\infty)$ Fock Space

For $\mathcal{N}=4$ SYM and the conifold, we saw that the multi-trace operator generating function was equal to the generating function for the Fock space for bosons moving on the transverse space. In the $\mathbb{C}^3/\mathbb{Z}_2$ theory, the extra branch of moduli space changes this story. Here, we have the generating function for the Fock space for bosons moving on

$\mathbb{C}^3/\mathbb{Z}_2$ multiplied by the generating function for bosons moving on $\mathbb{C}$:

$$F_M^{(\infty)}(a,b,c) = F_{\text{Fock}}(\mathbb{C}^3/\mathbb{Z}_2) \times F_{\text{Fock}}(\mathbb{C}). \tag{4.55}$$

This means that the multi-trace operator generating function gives the Fock space for bosons moving on $\mathbb{C}^3/\mathbb{Z}_2 \coprod \mathbb{C}$, where $\coprod$ indicates a disjoint union. So wavefunctions for the bosons can be any function in the space of functions on $\mathbb{C}^3/\mathbb{Z}_2$ or any function on the space of functions on $\mathbb{C}$, with only one identity function.

4.5.4 $N = 1$ Chiral Ring

In our previous examples, the generating function for the $U(1)$ chiral ring has been equal to the generating function for single-trace operators in the large N gauge theory. However, this is not the case in the present example. In the $U(\infty)$ theory, we cannot have $\text{Tr}(C_1^{m_1}C_2^{m_2})$ operators since C_1 and C_2 transform in different gauge groups; however when $N = 1$ these fields become uncharged so that the $C_1^{m_1}C_2^{m_2}$ operator is gauge-invariant. This means that we no longer have a one-to-one mapping between operators in the $U(1)$ theory and single-trace operators in the large N theory. The goal of this section is to derive the $N = 1$ chiral ring which includes the operators just mentioned.

We begin with the chiral ring of the $U(1)$ theory with no superpotential, which we denote $\mathcal{R}_0$. A basis for this ring is the set of gauge-invariant monomials built out of the fields W, X, Y, Z, C_1, C_2, $\mathcal{R}_0 = \{W^{n_1}X^{n_2}Y^{n_3}Z^{n_4}C_1^{n_5}C_2^{n_6} | n_i \geq 0\}$. To enforce the F-terms, we mod out by an ideal generated by the relevant constraints,

$$\mathcal{R} = \mathcal{R}_0/\mathcal{I}_0, \tag{4.56}$$

where $\mathcal{I}_0 = \langle X(C_1 - C_2), Y(C_1 - C_2), Z(C_1 - C_2), XY - Z^2, W - Z\rangle$. Thus the full chiral ring $\mathcal{R}$ is spanned by the basis

$$\{X^{n_1}Y^{n_2}Z^{n_3}C_1^{n_4}\} \cup \{C_1^{m_1}C_2^{m_2}\}, \tag{4.57}$$

where $n_1, n_2, n_4 \in \mathbb{Z}_{\geq 0}$, $n_3 \in \{0, 1\}$, $m_1 \in \mathbb{Z}_{\geq 0}$, and $m_2 \in \mathbb{Z}_+$ and the rule for multiplication is

$$\begin{aligned} (X^{n_1}Y^{n_2}Z^{n_3}C_1^{n_4}) \cdot (C_1^{m_1}C_2^{m_2}) &= X^{n_1}Y^{n_2}Z^{n_3}C_1^{n_4+m_1+m_2}, \quad n_1 + n_2 + n_3 > 0 \\ &= C_1^{n_4+m_1}C_2^{m_2}, \quad n_1 = n_2 = n_3 = 0 \end{aligned} \tag{4.58}$$

From this way of expressing the basis we see that this ring is $\mathcal{R}[\mathbb{C}^3/\mathbb{Z}_2 \cup \mathbb{C}^2]$, the ring of holomorphic polynomials on the space $\mathbb{C}^3/\mathbb{Z}_2 \cup \mathbb{C}^2$. The set of elements in this ring is the union of the set of functions on $\mathbb{C}^3/\mathbb{Z}_2$ and the set of functions on $\mathbb{C}^2$. These sets of functions have the bases $\{X^{n_1}Y^{n_2}Z^{n_3}C_1^{n_4}\}$ and $\{C_1^{m_1}C_2^{m_2}\}$, respectively, and share the coordinate C_1.

The generating function for the $U(1)$ chiral ring is then

$$F^{(1)}(x,y,z,c) = \sum_{n_4=0}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{1} c^{n_4}x^{n_1}y^{n_2}z^{n_3} + \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} c^{m_1+m_2} - \sum_{m=0}^{\infty} c^m. \quad (4.59)$$

We can see from this expression that the generating function is the sum of the generating functions for holomorphic polynomials on the two branches minus the generating function for holomorphic polynomials on the intersection. This subtraction is necessary to avoid double counting of operators.

We can further localise elements of $\mathcal{R}$ to the two branches of the moduli space by modding out by

$$\mathcal{I}_1 = \langle C_1 - C_2 \rangle = \{C_1^{m_1}(C_1^{m_2} - C_2^{m_2}) | m_1 \in \mathbb{Z}_{\geq 0}, m_2 \in \mathbb{Z}_+\} \quad (4.60)$$

for $\mathbb{C}^3/\mathbb{Z}_2$ or

$$\mathcal{I}_2 = \langle X, Y, Z \rangle = \{X^{n_1}Y^{n_2}Z^{n_3}C_1^{n_4} | n_i \in \mathbb{Z}_{\geq 0}, n_1 + n_2 + n_3 > 0\}. \quad (4.61)$$

for $\mathbb{C}^2$. The resulting quotient rings, $\mathcal{R}/\mathcal{I}_1$ and $\mathcal{R}/\mathcal{I}_2$, are identically the rings of holomorphic functions on the two branches. An alternate description of these two quotient rings of functions utilises minimal prime ideals[6]. Although the ideal $\mathcal{I}_0$ that we originally used to quotient $\mathcal{R}_0$ by is not a prime ideal, there exist two minimal prime ideals over $\mathcal{I}_0$, $\mathcal{I}'_1 = \langle C_1 - C_2, XY - Z^2, W - Z \rangle$ and $\mathcal{I}'_2 = \langle X, Y, Z, W - Z \rangle$. If we instead quotient $\mathcal{R}_0$ by $\mathcal{I}'_1$ or $\mathcal{I}'_2$ then we would have obtained $\mathcal{R}/\mathcal{I}_1$ and $\mathcal{R}/\mathcal{I}_2$, respectively. This is illustrated in figure 4.5.

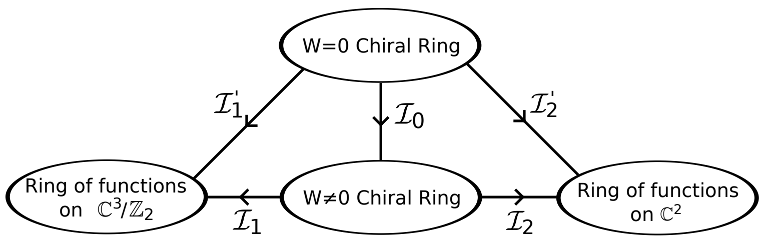

Figure 4.5: Chiral rings for the $\mathbb{C}^3/\mathbb{Z}_2$ theory. Following an arrow means quotienting by an ideal.

4.5.5 Conclusion

In this section we described how some of the relationships found in previous sections no longer hold. In particular:

1. The moduli space has two branches. One of these branches is the transverse space

[6] An ideal $\mathcal{I}$ is prime if $x \cdot y \in \mathcal{I}$ implies either $x \in \mathcal{I}$ or $y \in \mathcal{I}$, and a prime ideal $\mathcal{I}$ is a minimal prime ideal over $\mathcal{I}_0$ if there does not exist another prime ideal $\mathcal{I}'$ satisfying $\mathcal{I} \supset \mathcal{I}' \supset \mathcal{I}_0$.

$\mathbb{C}^3/\mathbb{Z}_2$ and the other is $\mathbb{C}^2$. This can be interpreted in terms of fractional branes as in [75].

2. The $U(1)$ chiral ring is equal to the ring of holomorphic polynomials on the moduli space and is slightly "larger" that the ring of holomorphic polynomials on the transverse space, in the sense that it contains the ring of holomorphic polynomials on the transverse space as a quotient ring.

3. The set of elements in the $U(1)$ chiral ring is different from the set of single-trace operators in the large N theory due to the presence of adjoint fields.

4. The multi-trace operator generating function is equal to the generating function of bosons moving on the space Transverse Space $\coprod \mathbb{C}$.

We will build on the results that we have found here in the coming sections and further elucidate these connections.

4.6 $\mathbb{C}^3/\mathbb{Z}_3$

We have seen some interesting phenomena in the $\mathbb{C}^3/\mathbb{Z}_2$ theory which do not quite fit in with the story that we had built up with the $\mathcal{N}=4$ SYM theory and the conifold theory. The question remains about what is special about the $\mathbb{C}^3/\mathbb{Z}_2$ theory. Is it the fact that this is an $\mathcal{N}=2$ theory? Or is it this something we will see with orbifold theories in general? In this section we look to examine the latter question by studying the $\mathbb{C}^3/\mathbb{Z}_3$ theory before going on in the next section to study the $\mathbb{C}^3/\mathbb{Z}_n$ in general. We will see that the $\mathbb{C}^3/\mathbb{Z}_3$ theory does not share the peculiarities that the $\mathbb{C}^3/\mathbb{Z}_2$ theory has.

The $\mathbb{C}^3/\mathbb{Z}_3$ theory is the world-volume gauge theory of N D3 branes probing a $\mathbb{C}^3/\mathbb{Z}_3$ singularity. From [74] we have that the theory is described by the $\mathcal{N}=1$ quiver diagram

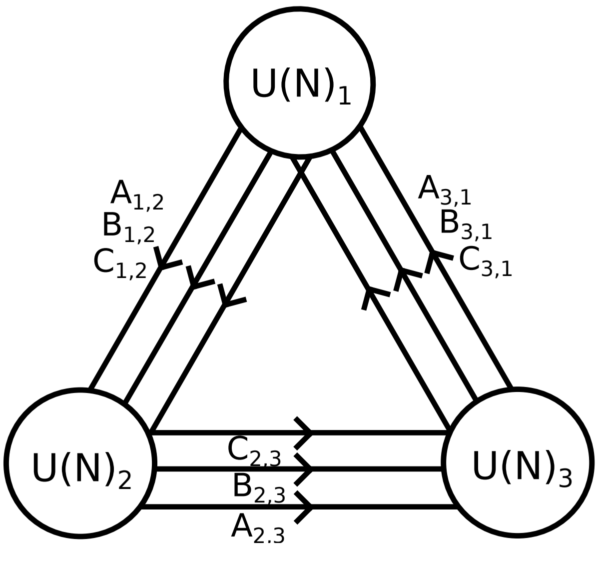

Figure 4.6: $\mathcal{N}=1$ quiver diagram for the $\mathbb{C}^3/\mathbb{Z}_3$ theory.

in figure 4.6 and that the superpotential is

$$W = \mathrm{Tr}\left[A_{1,2}\left(B_{2,3}C_{3,1} - C_{2,3}B_{3,1}\right) + B_{1,2}\left(C_{2,3}A_{3,1} - A_{2,3}C_{3,1}\right)\right. \\ \left. + C_{1,2}\left(A_{2,3}B_{3,1} - B_{2,3}A_{3,1}\right)\right] \tag{4.62}$$

and so the F-term relations are

$$\begin{array}{lll} B_{2,3}C_{3,1} = C_{2,3}B_{3,1}, & C_{2,3}A_{3,1} = A_{2,3}C_{3,1}, & A_{2,3}B_{3,1} = B_{2,3}A_{3,1}, \\ B_{3,1}C_{1,2} = C_{3,1}B_{1,2}, & C_{3,1}A_{1,2} = A_{3,1}C_{1,2}, & A_{3,1}B_{1,2} = B_{3,1}A_{1,2}, \\ B_{1,2}C_{2,3} = C_{1,2}B_{2,3}, & C_{1,2}A_{2,3} = A_{1,2}C_{2,3}, & A_{1,2}B_{2,3} = B_{1,2}A_{2,3}. \end{array} \tag{4.63}$$

4.6.1 $N = 1$ Moduli Space

The moduli space of this theory is parametrised by the following gauge-invariants

$$\begin{array}{llll} A_{1,2}A_{2,3}A_{3,1} & A_{1,2}A_{2,3}C_{3,1} & A_{1,2}C_{2,3}C_{3,1} & C_{1,2}C_{2,3}C_{3,1} \\ A_{1,2}A_{2,3}B_{3,1} & A_{1,2}B_{2,3}C_{3,1} & B_{1,2}C_{2,3}C_{3,1} & \\ A_{1,2}B_{2,3}B_{3,1} & B_{1,2}B_{2,3}C_{3,1} & & \\ B_{1,2}B_{2,3}B_{3,1} & & & . \end{array} \tag{4.64}$$

subject to relations that set products of these combinations to be equal if they have the same number of A's, B's and C's, e.g.

$$\left(A_{1,2}A_{2,3}A_{3,1}\right)\left(B_{1,2}B_{2,3}B_{3,1}\right) = \left(A_{1,2}A_{2,3}B_{3,1}\right)\left(A_{1,2}B_{2,3}B_{3,1}\right). \tag{4.65}$$

This is simply the space $\mathbb{C}^3/\mathbb{Z}_3$. To see this consider taking $\mathbb{C}^3$ with co-ordinates (x, y, z) and quotient by $\mathbb{Z}_3$ which has the following action on the co-ordinates:

$$\mathbb{Z}_n = \left\{ \begin{pmatrix} \omega_3^k & & \\ & \omega_3^k & \\ & & \omega_3^k \end{pmatrix}, 1 \le k \le n \right\}. \tag{4.66}$$

Then the $\mathbb{Z}_3$ invariant co-ordinates which parametrise the $\mathbb{C}^3/\mathbb{Z}_3$ space are x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z, xz^2, yz^2, z^3 subject to the relations, e.g. $(x^3)(y^3) = (x^2y)(xy^2)$. We can see from this that there is a one-to-one mapping to the co-ordinates on our moduli space where the mapping is roughly $x \sim A$, $y \sim B$, $z \sim C$.

In contrast to the previous section there is only one branch on the moduli space and the moduli space is the same as the transverse space as was the case for $\mathcal{N} = 4$ SYM and the conifold theory.

4.6.2 $W = 0$ Large N Chiral Ring

For the $W = 0$ large N theory all single trace operators have $3n$ letters. The single trace operator generating function is the generating function for 27-ary necklaces of beads. The 27 colours correspond to $A_{1,2}A_{2,3}A_{3,1}$, $A_{1,2}A_{2,3}B_{3,1}$, etc. The function is

$$\boxed{F_S(x_1, ..., x_{27}) = -\sum_{d=1}^{\infty} \frac{\varphi(d)}{d} \log\left[1 - \left(x_1^d + x_2^d + \cdots + x_{27}^d\right)\right]}. \tag{4.67}$$

The plethystic exponential gives us the multi-trace operator generating function

$$\boxed{F_M(x_1, ..., x_{27}) = \prod_{n=1}^{\infty} \frac{1}{1 - (x_1^n + x_2^n + \cdots + x_{27}^n)}}. \tag{4.68}$$

4.6.3 $W \neq 0$ Large N Chiral Ring

For the $W \neq 0$ large N theory again all single-trace operators must have $3n$ letters. The F-term relations conserve three quantities: the number A's, the number of B's and the number of C's. So our generating function will have three chemical potentials. If we have an operator with n_1 A operators, n_2 B operators and $3n - n_1 - n_2$ C operators then the F-terms tell us that it is equivalent to any other operator with n_1 A operators, n_2 B operators and $3n - n_1 - n_2$ C operators. So if we take the generating function for the $W = 0$ case and set $a_{1,2} = a_{2,3} = a_{3,1} = a$, $b_{1,2} = b_{2,3} = b_{3,1} = b$, $c_{1,2} = c_{2,3} = c_{3,1} = c$ then it is of the form

$$F_S(a, b, c) = \sum_{n=0}^{\infty} \sum_{n_1=0}^{3n} \sum_{n_2=0}^{3n-n_1} k_{n_1,n_2,n_3} a^{n_1} b^{n_2} c^{3n-n_1-n_2}. \tag{4.69}$$

Then the generating function for $W \neq 0$ is obtained by setting all $k_{n_1,n_2,n_3} = 1$:

$$\boxed{F_S(a, b, c) = \sum_{n=0}^{\infty} \sum_{n_1=0}^{3n} \sum_{n_2=0}^{3n-n_1} a^{n_1} b^{n_2} c^{3n-n_1-n_2}}. \tag{4.70}$$

This has the closed form

$$F_S(a, b, c) = \frac{a^2\left(b^2c^2 + b + c\right) + a\left(b^2 + bc + c^2\right) + b^2c + bc^2 + 1}{(1 - a^3)(1 - b^3)(1 - c^3)}. \tag{4.71}$$

When we make the substitution $a \to t, b \to t, c \to t$ we recover the formula from [70]:

$$\frac{1 + 7t^3 + t^6}{(1 - t^3)^3}. \tag{4.72}$$

Taking the plethystic exponential we get the generating function for multi-trace operators

$$\boxed{F_M\left(a,b,c\right) = \prod_{n=0}^{\infty}\prod_{n_1=0}^{3n}\prod_{n_2=0}^{3n-n_1}\frac{1}{1-a^{n_1}b^{n_2}c^{3n-n_1-n_2}}}. \tag{4.73}$$

4.6.4 $U(\infty)$ Fock Space

The Fock space indicated by equation (4.73) is simply the Fock space of bosons on $\mathbb{C}^3/\mathbb{Z}_3$. This means that the multi-trace operator generating function is equal to the generating function for the Fock space of bosons moving on the transverse space, $\mathbb{C}^3/\mathbb{Z}_3$, and so fits in with the story that we built up with $\mathcal{N}=4$ SYM and the conifold theory.

4.6.5 $N=1$ Chiral Ring

The generating function for the $N=1$ chiral ring for this theory is given by equation (4.70). This is because there is a one-to-one mapping between gauge-invariant operators in the $U(1)$ theory and single-trace operators in the large N theory. This mapping is simply

$$A^{n_1}B^{n_2}C^{n_3} \to \mathrm{Tr}(A^{n_1}B^{n_2}C^{n_3}) \tag{4.74}$$

This means that the $U(1)$ chiral ring is just the ring of holomorphic polynomials on $\mathbb{C}^3/\mathbb{Z}_3$ and so this fits in the with the story that we were originally seeing with the $\mathcal{N}=4$ SYM theory and the conifold theory.

4.6.6 Conclusion

In order to try and answer the question of whether the peculiarities of the $\mathbb{C}^3/\mathbb{Z}_2$ theory were common to the family of $\mathbb{C}^3/\mathbb{Z}_n$ theories, or indeed orbifold theories in general, we examined the $\mathbb{C}^3/\mathbb{Z}_3$ theory. We found that in fact it seems to fit in with the story that was already emerging with the $\mathcal{N}=4$ SYM theory and the conifold theory. The $N=1$ moduli space is the same as the space transverse to the branes, the $N=1$ chiral ring is the ring of holomorphic polynomials on the transverse space and the multi-trace operator generating function is the generating function for the Fock space of bosons moving on the transverse space.

It seems that the $\mathbb{C}^3/\mathbb{Z}_2$ theory was in fact a peculiarity in the $\mathbb{C}^3/\mathbb{Z}_n$ family of theories however we will see in the next section that there are actually more theories in the $\mathbb{C}^3/\mathbb{Z}_n$ family that exhibit the characteristics of the $\mathbb{C}^3/\mathbb{Z}_2$ theory.

4.7 $\mathbb{C}^3/\mathbb{Z}_n$

We now consider the world-volume gauge theory of N D3 branes probing a $\mathbb{C}^3/\mathbb{Z}_n$ singularity, with the $\mathbb{Z}_n$ action on the coordinates of $\mathbb{C}^3$ given by

$$\mathbb{Z}_n = \left\{ \begin{pmatrix} \omega_n^k & & \\ & \omega_n^k & \\ & & \omega_n^{-2k} \end{pmatrix}, 1 \leq k \leq n \right\}. \tag{4.75}$$

as studied in [74]. This theory has the quiver diagram in figure 4.7 and superpotential

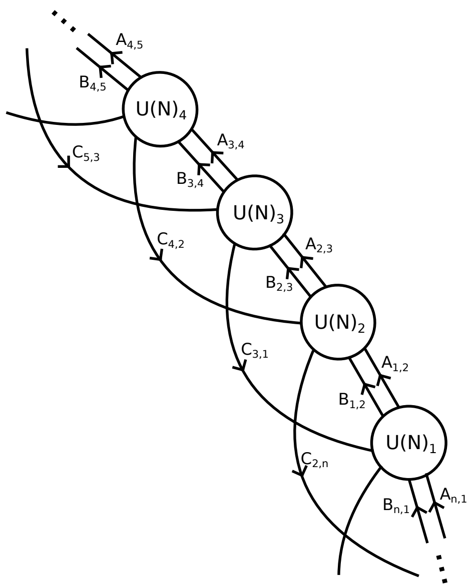

Figure 4.7: $\mathcal{N} = 1$ quiver diagram for the $\mathbb{C}^3/\mathbb{Z}_n$ theory. The quiver is a circle of n nodes.

$$W = \sum_{i=1}^{n} \left(A_{i,i+1}B_{i+1,i+2} - B_{i,i+1}A_{i+1,i+2}\right) C_{i+2,i}. \tag{4.76}$$

The F-term relations are

$$\begin{aligned} A_{i,i+1}B_{i+1,i+2} &= B_{i,i+1}A_{i+1,i+2}, \\ B_{i+1,i+2}C_{i+2,i} &= C_{i+1,i-1}B_{i-1,i}, \\ A_{i+1,i+2}C_{i+2,i} &= C_{i+1,i-1}A_{i-1,i}. \end{aligned} \tag{4.77}$$

4.7.1 $N = 1$ Moduli Space

To find the moduli space of this theory, we work with a set of gauge-invariant monomials similar to those of the previous section. However, due to the structure of the quiver, we must use slightly different sets for even and odd n. Useful coordinates are given by

$$X_{a,b,c} = \prod_{i=1}^{a} A_{i,i+1} \prod_{j=1}^{b} B_{a+j,a+j+1} \prod_{k=1}^{c} C_{a+b+1+(k-1)(n-2),a+b+1+k(n-2)} \tag{4.78}$$

so that essentially, $X_{a,b,c} \sim A^a B^b C^c$ and all subscripts are modulo n. These parameters are subject to the relation $X_{a_1,b_1,c_1} X_{a_2,b_2,c_2} = X_{a_1+a_2,b_1+b_2,c_1+c_2}$. The gauge-invariants for n odd are

$$\begin{array}{cccc} X_{n,0,0} & X_{2,0,1} & X_{1,0,\frac{n+1}{2}} & X_{0,0,n} \\ X_{n-1,1,0} & X_{1,1,1} & X_{0,1,\frac{n+1}{2}} & \\ \vdots & X_{0,2,1} & & \\ X_{0,n,0} & & & \end{array}$$

The space spanned by these coordinates subject to $X_{a_1,b_1,c_1} X_{a_2,b_2,c_2} = X_{a_1+a_2,b_1+b_2,c_1+c_2}$ is the space $\mathbb{C}^3/\mathbb{Z}_n$ with odd n. For even n, we use

$$\begin{array}{ccc} X_{n,0,0} & X_{2,0,1} & X_{0,0,n/2} \\ X_{n-1,1,0} & X_{1,1,1} & Y_{0,0,n/2} \\ \vdots & X_{0,2,1} & \\ X_{0,n,0} & & \end{array}$$

where $X_{a,b,c}$ is as above, and we have added $X_{0,0,n/2} = C_{1,n-1} \ldots C_{3,1}$ and $Y_{0,0,n/2} = C_{2,n} \ldots C_{4,2}$. There is now an additional relation $X_{a,b,c} Y^m_{0,0,n/2} \sim X_{a,b,c+nm/2}$ when $a+b > 0$.

The additional coordinate $Y_{0,0,n/2}$ is required to account for the fact that there are two distinct gauge-invariant operators with $n/2$ C's. The moduli space for even n has two different branches:

1. $\{X_{n,0,0}, \ldots, X_{0,n,0}, X_{2,0,1}, X_{1,1,1}, X_{0,2,1}, X_{0,0,n/2}, Y_{0,0,n/2}\}$ subject to $X_{a_1,b_1,c_1} X_{a_2,b_2,c_2} = X_{a_1+a_2,b_1+b_2,c_1+c_2}$ and $X_{0,0,n/2} = Y_{0,0,n/2}$,
2. $\{X_{n,0,0}, \ldots, X_{0,n,0}, X_{2,0,1}, X_{1,1,1}, X_{0,2,1}, X_{0,0,n/2}, Y_{0,0,n/2}\}$ subject to $X_{a,b,c} = 0$ when $a+b > 0$.

We see that, as with the $\mathbb{C}^3/\mathbb{Z}_2$ theory, there are multiple branches of moduli space. Specifically, there is one main branch of the moduli space which is $\mathbb{C}^3/\mathbb{Z}_n$ and an extra branch which is $\mathbb{C}^2$, with coordinates $X_{0,0,n/2}$ and $Y_{0,0,n/2}$. We denote this full moduli space $\mathbb{C}^3/\mathbb{Z}_n \cup \mathbb{C}^2$, where the union is defined so that the spaces share the line defined by $X_{0,0,n/2} = Y_{0,0,n/2}$ and $X_{a,b,c} = 0$ for $a+b > 0$.

4.7.2 $W = 0$ Large N Chiral Ring

Using the Pólya enumeration theorem, the generating function for single-trace operators is of the form

$$F_S^{(\infty)}(x_i) = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log\left[1 - f\left(x_i^k\right)\right] \tag{4.79}$$

and the multi-trace operator generating function is of the form

$$F_M^{(\infty)}(x_i) = \prod_{k=1}^{\infty} \frac{1}{1 - f\left(x_i^k\right)}. \tag{4.80}$$

for some colour generating function $f(x_i)$. This function has a term for every closed loop in the quiver. In combinatorics language, each closed loop constitutes a colour. We can make necklaces (single-trace operators) by combining beads of different colours (closed loops of operators) in a necklace (trace). However, in our colour generating function we must subtract the product of any two loops that do not overlap and thus can not be placed beside each other in a necklace. This means that we subtract the contributions from operators where there is a product over two loops that do not share a node.

Similarly, it is necessary add terms to $f\left(x_i^k\right)$ that are cubic in non-intersecting loops, subtract terms that are quartic, and so on. As an example, consider the colour generating function for 3-ary necklaces of beads where none of the colours can be placed beside each other. This is simply the sum of three generating functions of 1-ary necklaces:

$$\begin{aligned} F_S^{(\infty)}(x,y,z) &= -\sum_{n=1}^{\infty} \frac{\varphi(d)}{d} \log\left[1 - x^n\right] + \log\left[1 - y^n\right] + \log\left[1 - z^n\right] \\ &= -\sum_{n=1}^{\infty} \frac{\varphi(d)}{d} \log\left[1 - \left(x^n + y^n + z^n - x^n y^n - x^n z^n - y^n z^n + x^n y^n z^n\right)\right]. \end{aligned} \tag{4.81}$$

The colour generating function is then

$$f(x,y,z) = x + y + z - xy - xz - yz + xyz. \tag{4.82}$$

As we go to higher number of colours, we must continue this pattern of addition and subtraction. This is in agreement with the formula found in [90]:

$$F_M^{(\infty)}(x,y,z) = \text{P.E.}[F_S^{(\infty)}(x,y,z)] = \prod_i \frac{1}{\det(\mathbb{I} - X_n(x^i, y^i, z^i))}, \tag{4.83}$$

where X_n is the weighted adjacency matrix for the graph described by the quiver.

4.7.3 $W \neq 0$ Large N Chiral Ring

We now describe the generating functions for even and odd n. As these derivations are rather lengthy, we relegate them to appendix A.5 and here simply summarise the results.

For general odd n the generating function is

$$\boxed{F_S^{(\infty)}(a,b,c) = \left[\sum_{m=0}^{\infty} c^{nm}\right]\left[\sum_{j=0}^{\frac{n-1}{2}}\sum_{\ell=0}^{\infty}\sum_{k=0}^{n\ell+2j} a^k b^{n\ell+2j-k} c^j + c^{\frac{n+1}{2}}\sum_{j=0}^{\frac{n-3}{2}}\sum_{\ell=0}^{\infty}\sum_{k=0}^{n\ell+2j+1} a^k b^{n\ell+2j+1-k} c^j\right]}. \tag{4.84}$$

which has the rational form

$$\begin{aligned} F_S^{(\infty)}(a,b,c) =& \frac{1}{b-a}\frac{1}{1-c^n}\left[\frac{b\left(-b^n c^n - \left(b^2 c\right)^{\frac{n+1}{2}} + bc^{\frac{n+1}{2}} + 1\right)}{\left(1-b^2 c\right)\left(1-b^n\right)}\right. \\ & \left. - \frac{a\left(-a^n c^n - \left(a^2 c\right)^{\frac{n+1}{2}} + ac^{\frac{n+1}{2}} + 1\right)}{\left(1-a^2 c\right)\left(1-a^n\right)}\right]. \end{aligned} \tag{4.85}$$

We can then get the multi-trace operator generating function using plethystics:

$$\boxed{F_M^{(\infty)}(a,b,c) = \prod_{m=0}^{\infty}\left[\prod_{j=0}^{\frac{n-1}{2}}\prod_{\ell=0}^{\infty}\prod_{k=0}^{n\ell+2j}\frac{1}{1-c^{nm+j}a^k b^{n\ell+2j-k}}\right]\left[\prod_{j=0}^{\frac{n-3}{2}}\prod_{\ell=0}^{\infty}\prod_{k=0}^{n\ell+2j+1}\frac{1}{1-c^{nm+\frac{n+1}{2}+j}a^k b^{n\ell+2j-k}}\right]} \tag{4.86}$$

Taking $a \to t$, $b \to t$, $c \to t$ in equation (4.84), we get

$$F_S^{(\infty)}(t) = \frac{-t^{2n} - nt^{n+3} - t^{2n+3} - 2t^{\frac{3(n+1)}{2}} + 2t^{\frac{n+3}{2}} + nt^n + t^3 + 1}{\left(t^3-1\right)^2\left(t^n-1\right)^2}, \tag{4.87}$$

which agrees with the result which can be calculated using the methods in [70] for general odd n.

For even n the generating function is

$$\boxed{F_S^{(\infty)}(a,b,c) = \sum_{m=0}^{\infty} c^{\frac{nm}{2}}\sum_{j=0}^{\frac{n}{2}-1}\sum_{\ell=0}^{\infty}\sum_{k=0}^{n\ell+2j} a^{n\ell+2j-k} b^k c^j + \sum_{m=1}^{\infty} c^{\frac{nm}{2}}}, \tag{4.88}$$

which has the rational form

$$F_S^{(\infty)}(a,b,c) = \frac{1}{(a-b)\left(1-c^{n/2}\right)}\left[\frac{a\left(1-\left(a^2c\right)^{n/2}\right)}{\left(1-a^2c\right)\left(1-a^n\right)} - \frac{b\left(1-\left(b^2c\right)^{n/2}\right)}{\left(1-b^2c\right)\left(1-b^n\right)}\right] + \frac{c^{n/2}}{1-c^{n/2}}. \tag{4.89}$$

The multi-trace operator generating function is then

$$\boxed{F_M^{(\infty)}(a,b,c) = \left[\prod_{m=0}^{\infty}\prod_{j=0}^{\frac{n}{2}-1}\prod_{\ell=0}^{\infty}\prod_{k=0}^{n\ell+2j}\frac{1}{1-a^{n\ell+2j-k}b^k c^{j+\frac{nm}{2}}}\right]\left[\prod_{m=1}^{\infty}\frac{1}{1-c^{\frac{nm}{2}}}\right]}. \tag{4.90}$$

Taking $a \to t$, $b \to t$, $c \to t$ in equation (4.88) yields

$$F_S^{(\infty)}(t) = \frac{\frac{n\left(1-\left(t^3\right)^{n/2}\right)t^n}{(1-t^3)(1-t^n)^2} + \frac{1-\left(t^3\right)^{n/2}}{(1-t^3)(1-t^n)} + \frac{2\left(-\frac{1}{2}nt^{3n/2}+\left(\frac{n}{2}-1\right)t^{\frac{3n}{2}+3}+t^3\right)}{(1-t^3)^2(1-t^n)}}{1-t^{n/2}} + \frac{t^{n/2}}{1-t^{n/2}}. \quad (4.91)$$

While the first term is the one that can be found using the methods in [70], the second term is new.

$U(\infty)$ **Fock Space**

We can see from equation (4.90) that the generating function for multi-trace operators in the large N $\mathbb{C}^3/\mathbb{Z}_n$ theory with even n is equal to the generating function for the Fock space of bosons on the transverse space times an extra factor. More specifically, it is

$$F_M^{(\infty)}(a,b,c) = F_{\text{Fock}}(\mathbb{C}^3/\mathbb{Z}_n) \times F_{\text{Fock}}(\mathbb{C}). \quad (4.92)$$

Similarly to the $\mathbb{C}^3/\mathbb{Z}_2$ theory, this is the generating function for the Fock space of bosons moving on $\mathbb{C}^3/\mathbb{Z}_n \coprod \mathbb{C}$, where $\coprod$ indicates a disjoint union. For odd n we do not have this extra factor, and the multi-trace operator generating function is equal to the generating function for the Fock space on $\mathbb{C}^3/\mathbb{Z}_n$.

4.7.4 $N=1$ Chiral Ring

For the case of odd n there is a one-to-one mapping between operators in the $U(1)$ theory and single-trace operators in the large N theory; this mapping is $X_{a,b,c} \to \text{Tr}(X_{a,b,c})$. Because of this the generating function for the $U(1)$ chiral ring for this theory is simply the generating function for single-trace operators in the large N theory, given in equation (4.84). Here the chiral ring has the basis

$$\{X_{a,b,c} | a+b+c(n-2) \equiv 0 \mod n\}, \quad (4.93)$$

with $X_{a,b,c}$ as in section 4.7.1. This is the ring generated by the gauge-invariant operators

$$\begin{array}{cccc} X_{n,0,0} & X_{2,0,1} & X_{1,0,\frac{n+1}{2}} & X_{0,0,n} \\ X_{n-1,1,0} & X_{1,1,1} & X_{0,1,\frac{n+1}{2}} & \\ \vdots & X_{0,2,1} & & \\ X_{0,n,0} & & & \end{array}$$

For even n, things are slightly different. The set of elements in the $U(1)$ chiral ring for this theory is not equal to the set of single-trace operators in the large N theory; this is because for $N>1$ the $X_{0,0,n/2}$ and $Y_{0,0,n/2}$ operators transform in the adjoint representations of the $U(N)_1$ and $U(N)_2$ gauge groups, respectively. Thus we cannot have single-trace operators of the form $\text{Tr}(X_{0,0,n/2}^{m_1} Y_{0,0,n/2}^{m_2})$ in the large N theory, though in the $U(1)$ theory operators of the form $X_{0,0,n/2}^{m_1} Y_{0,0,n/2}^{m_2}$ are allowed. This means that there is *not* a one-to-one mapping between operators in the $U(1)$ theory and single-trace operators in the large N theory.

For even n, the $U(1)$ chiral ring has the basis

$$\{X_{a,b,c}|a+b+c(n-2)\equiv 0 \mod n\}\cup\{X_{0,0,nm_1/2}Y_{0,0,nm_2/2}|m_1\in\mathbb{Z}_{\geq 0},m_2\in\mathbb{Z}_+\},\quad(4.94)$$

modulo the equivalence $X_{a_1,b_1,c_1}X_{a_2,b_2,c_2}\sim X_{a_1+a_2,b_1+b_2,c_1+c_2}$ and $X_{a,b,c}Y_{0,0,nm/2}\sim X_{a,b,c+nm/2}$ when $a+b>0$, with $Y_{0,0,nm/2}=Y^m_{0,0,n/2}$ and $X_{a,b,c}$ and $Y_{0,0,n/2}$ as defined in section 4.7.1. Again we see that the chiral ring is equal to the ring of holomorphic polynomials on the moduli space, $\mathbb{C}^3/\mathbb{Z}_n\cup\mathbb{C}^2$ and the counting function is given by

$$F^{(1)}(a,b,c)=\sum_{m=0}^{\infty}\sum_{j=0}^{\frac{n}{2}-1}\sum_{\ell=0}^{\infty}\sum_{k=0}^{n\ell+2j}a^{n\ell+2j-k}b^kc^{j+\frac{nm}{2}}+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}c^{\frac{n}{2}(m_1+m_2)}-\sum_{m=0}^{\infty}c^{\frac{n}{2}m}.\quad(4.95)$$

As was the case in the $\mathbb{C}^3/\mathbb{Z}_2$ example, this generating function is a sum of three terms. The first is the generating function for the ring of holomorphic polynomials on $\mathbb{C}^3/\mathbb{Z}_n$. The second is the generating function for the ring of holomorphic polynomials on $\mathbb{C}^2$. Finally, the last term subtracts the generating function for the ring of holomorphic polynomials on the intersection. This generating function then is the generating function for holomorphic polynomials on the moduli space $\mathbb{C}^3/\mathbb{Z}_n\cup\mathbb{C}^2$.

As was the case with the $\mathbb{C}^3/\mathbb{Z}_2$ theory we can go from the chiral ring, $\mathcal{R}=\mathcal{R}_0/\mathcal{I}_0$, to the ring of holomorphic polynomials on the two branches of moduli space by quotienting by two ideals. In the case of the $\mathbb{C}^3/\mathbb{Z}_n$ theory with even n the two ideals are $\mathcal{I}_1=\langle(X_{0,0,n/2}-Y_{0,0,n/2})\rangle$ and $\mathcal{I}_2=\langle X_{n,0,0},X_{n-1,1,0},\ldots,X_{0,n,0},X_{2,0,1},X_{1,1,1},X_{0,2,1}\rangle$. Alternatively, we could have obtained these two rings from the $W=0$ chiral ring using minimal prime ideals. The ideal $\mathcal{I}_0$ is not a prime ideal and there are two minimal prime ideals, $\mathcal{I}'_1$ and $\mathcal{I}'_2$ over $\mathcal{I}_0$. Quotienting $\mathcal{R}_0$ by these two ideals gives the ring of functions on $\mathbb{C}^3/\mathbb{Z}_n$ and $\mathbb{C}^2$. This is analogous to the situation depicted in figure 4.5.

4.7.5 Conclusion

In this section we have attempted to find out how the $\mathbb{C}^3/\mathbb{Z}_n$ theory fits into our description of moduli spaces, chiral rings, and Fock spaces of branes. For n odd, the $\mathbb{C}^3/\mathbb{Z}_n$ theory fits into this story in the same way as $\mathcal{N}=4$ and the conifold. The moduli space is simply $\mathbb{C}^3/\mathbb{Z}_n$ and the set of elements in the $U(1)$ chiral ring is the set of functions on $\mathbb{C}^3/\mathbb{Z}_n$. This set is equal to the set of single-trace operators, and the multi-trace operator generating function is equal to the generating function for the Fock space of bosons moving on $\mathbb{C}^3/\mathbb{Z}_n$.

For n even, the story needs to be amended slightly:

1. The moduli space has one main branch where the space is simply $\mathbb{C}^3/\mathbb{Z}_n$, however it also has an extra branch where the space is $\mathbb{C}^2$.

2. The set of elements in the $U(1)$ chiral ring is then not quite the set of functions on $\mathbb{C}^3/\mathbb{Z}_n$, but rather the set of functions on $\mathbb{C}^3/\mathbb{Z}_n \cup \mathbb{C}^2$.

3. The set of elements in the $U(1)$ chiral ring is also not equal to the set of single-trace operators in the large N theory.

4. The multi-trace operator generating function is equal to the generating function for the Fock space for bosons on $\mathbb{C}^3/\mathbb{Z}_n \coprod \mathbb{C}$.

4.8 $\mathbb{C}^3/\hat{A}_n$

We now consider the $\mathbb{C}^3/\hat{A}_n$ theory studied in [59, 91], with the action of $\hat{A}_n$ defined by

$$\mathbb{Z}_n = \left\{ \begin{pmatrix} \omega_n^k & & \\ & \omega_n^{-k} & \\ & & 1 \end{pmatrix}, 1 \leq k \leq n \right\}. \tag{4.96}$$

The $\mathcal{N} = 1$ quiver diagram for this theory is as shown in figure 4.8, and the superpotential

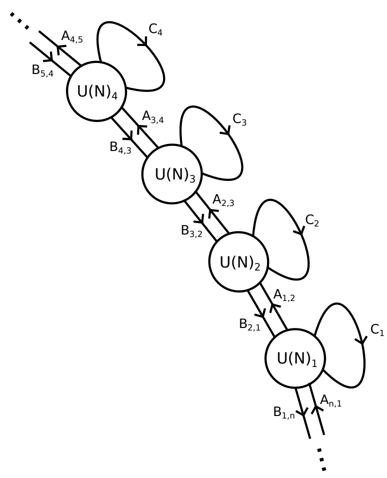

Figure 4.8: The $\mathcal{N} = 1$ quiver diagram for the the $\mathbb{C}^3/\hat{A}_n$ theory.

is

$$W = \sum_{i=1}^{n} C_i \left(A_{i,i+1} B_{i+1,i} - B_{i,i-1} A_{i,i-1} \right). \tag{4.97}$$

The F-term relations are

$$\begin{aligned} A_{i,i+1} B_{i+1,i} &= B_{i,i-1} A_{i-1,i}, \\ B_{i+1,i} C_i &= C_{i+1} B_{i+1,i}, \\ A_{i,i+1} C_{i+1} &= C_i A_{i,i+1}. \end{aligned} \tag{4.98}$$

The problem of counting multi-trace operators for this theory along with other $\mathcal{N} = 2$ theories was also considered in [75] where they derive the Higgs and Coulomb branch generating functions separately and then combine them.

4.8.1 $N = 1$ Moduli Space

The moduli space of this theory is parametrised by the operators $X = A_{1,2}A_{2,3}\dots A_{n,1}$, $Y = B_{1,n}B_{n,n-1}\dots B_{2,1}$, $Z = A_{1,2}B_{2,1}$, C_1,..., C_{n-1}, and C_n, subject to the relation $XY = Z^n$. As was the case for the $\mathbb{C}^3/\mathbb{Z}_n$ theory with even n, there are two branches of solutions:

1. $\{X, Y, Z, C_1, \dots, C_n\}$ subject to $XY = Z^n$, $C_1 = C_2 = \dots = C_n$.
2. $\{X, Y, Z, C_1, \dots, C_n\}$ subject to $X = Y = Z = 0$.

The first branch is simply $\mathbb{C}^3/\hat{A}_n$, and the second branch is $\mathbb{C}^n$. Again we see the presence of one main branch identical to the transverse space to the D3-branes as well as an extra branch. We denote this space as $\mathbb{C}^3/\hat{A}_n \cup \mathbb{C}^n$, where the union is such that the two branches share the line $X = Y = Z = 0$, $C_1 = C_2 = \dots = C_n$. The picture for this moduli space is as in figure 4.4, except that now the extra branch is n-dimensional.

The first branch of this moduli space is a mixed branch since it contains both Higgs and Coulomb branch operators. The second branch however is a purely Coulomb branch. The first branch contains as a sub-space the $C_i = 0$, pure Higgs branch.

4.8.2 $W = 0$ Large N Chiral Ring

As in the previous section the single- and multi-trace operator generating functions are

$$F_S^{(\infty)}(x_i) = -\sum_{n=1}^{\infty} \frac{\varphi(d)}{d} \log\left[1 - f\left(x_i^n\right)\right],$$
$$F_M^{(\infty)}(x_i) = \prod_{n=1}^{\infty} \frac{1}{1 - f\left(x_i^n\right)}, \tag{4.99}$$

where the colour generating function is determined in the previous section, matching the results of [90].

4.8.3 $W \neq 0$ Large N Chiral Ring

When we have a non-zero superpotential the F-term relations tell us that any single-trace operator with n_1 A's, n_2 B's and n_3 C's is equivalent to any other single-trace operator with n_1 A's, n_2 B's and n_3 C's. The only exception to this rule is when $n_1 = n_2 = 0$. In that case there are n different operators we can have of the form $\mathrm{Tr}(C_i^{n_3})$. For the first set of operators we can construct any gauge-invariant operator by inserting $A_{1,2}B_{2,1}$, $A_{1,2}\dots A_{n,1}$, $B_{1,n}, \dots B_{2,1}$ and C_1 into a trace recursively. This means that the generating function will be

$$\boxed{F_S^{(\infty)}(a,b,c) = \sum_{m=0}^{n-1}\sum_{j=0}^{\infty} a^{nj+m}\sum_{k=0}^{\infty} c^k \sum_{l=0}^{\infty} b^{nl+m} + (n-1)\sum_{k=1}^{\infty} c^k}, \tag{4.100}$$

with rational form

$$F_S^{(\infty)}(a,b,c) = \frac{1-(ab)^n}{(1-a^n)(1-b^n)(1-c)(1-ab)} + \frac{(n-1)c}{1-c}. \tag{4.101}$$

Taking $a, b, c \to t$ gives

$$F_S^{(\infty)}(t) = \frac{1+t^n}{(1-t^n)(1-t)(1-t^2)} + \frac{(n-1)t}{1-t^n}. \tag{4.102}$$

The first term of this formula matches the result given in [70]. The presence of the second term is caused by considering the operators $\mathrm{Tr}(C_i^k)$ and $\mathrm{Tr}(C_j^k)$ to be inequivalent for $i \neq j$ and is considered in [75].

We can use plethystics once again to get the multi-trace operator generating function:

$$\boxed{F_M^{(\infty)}(a,b,c) = \left[\prod_{m=0}^{n-1}\prod_{j=0}^{\infty}\prod_{k=0}^{\infty}\prod_{l=0}^{\infty}\frac{1}{1-a^{nj+m}b^{nl+m}c^k}\right]\left[\sum_{k=1}^{\infty}\frac{1}{1-c^k}\right]^{n-1}}. \tag{4.103}$$

$U(\infty)$ **Fock Space**

From equation (4.100) one can see that the large N set of single-trace operators is not equal to set of elements in the ring of holomorphic polynomials on $\mathbb{C}^3/\hat{A}_n$. As a consequence of this, the multi-trace operator generating function is not equal to the generating function for the Fock space of bosons on $\mathbb{C}^3/\hat{A}_n$. Instead, as can be seen from equation (4.103) we have

$$F_M^{(\infty)}(a,b,c) = F_{\mathrm{Fock}}(\mathbb{C}^3/\hat{A}_n) \times [F_{\mathrm{Fock}}(\mathbb{C})]^{n-1}. \tag{4.104}$$

This is the generating function for the multi-particle Fock space of bosons moving on $\mathbb{C}^3/\hat{A}_n\,[\coprod\mathbb{C}]^{n-1}$, *i.e.*, it is the Fock space for bosons that can have wavefunctions either in the ring of holomorphic polynomials on $\mathbb{C}^3/\hat{A}_n$ or in the ring of holomorphic polynomials on any of $n-1$ copies of $\mathbb{C}$.

4.8.4 $N = 1$ Chiral Ring

As was the case in the previous two sections the set of elements in the $N = 1$ chiral ring for the $\mathbb{C}^3/\hat{A}_n$ theory differs from the set of single trace operators in the large N theory. This is because there are operators of the form $C_1^{k_1}\dots C_n^{k_n}$ in the $N = 1$ theory whose analogue in the set of large N single-trace operators, $\mathrm{Tr}(C_1^{k_1}\dots C_n^{k_n})$ do not exist.

So, the chiral ring for the $\mathbb{C}^3/\hat{A}_n$ theory has the following basis:

$$\{X^{n_1}Y^{n_2}Z^{n_3}C_1^{n_4}|n_{1,2,4} \in \mathbb{Z}_{\geq 0}, n_3 \in [0,n-1]\cap\mathbb{Z}\}\cup\{C_1^{m_1}C_2^{m_2}\cdots C_n^{m_n}|m_i \in \mathbb{Z}_{\geq 0}, \sum_{i=2}^{n} m_i > 0\}, \tag{4.105}$$

modulo the relation $X^{n_1}Y^{n_2}Z^{n_3}C_1^{m_4}C_2^{m_2}\cdots C_n^{m_n} \sim X^{n_1}Y^{n_2}Z^{n_3}C_1^{\sum_i m_i}$ when $n_1 + n_2 + n_3 > 0$. The counting function for this ring is

$$F^{(1)}(x,y,z,c) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_3=0}^{n-1} x^{n_1}y^{n_2}z^{n_3}c^{n_4} + \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} c^{m_1+\ldots m_n} - \sum_{m=1}^{\infty} c^m. \tag{4.106}$$

From this we can see once again that the generating function for the $N = 1$ theory is the sum of the generating functions for the two branches minus the generating function for the intersection.This also matches the result found in [75].

We can go from the $N = 1$ chiral ring, $\mathcal{R}$, to the ring of functions on either of the two branches by quotienting by the ideals

$$\mathcal{I}_1 = \langle (C_2-C_1), (C_3-C_1), \cdots, (C_n-C_1)\rangle = \{C_1^{m_1}\cdots C_n^{m_n} - C_1^{\sum_{i=1}^n m_i}|m_i \in \mathbb{Z}_{\geq 0}\}. \tag{4.107}$$

for $\mathbb{C}^3/\hat{A}_n$ and

$$\mathcal{I}_2 = \langle X, Y, Z\rangle = \{X^{n_1}Y^{n_2}Z^{n_3}C_1^{n_4}|n_{1,2,4} \in \mathbb{Z}_{\geq 0}, n_3 \in [0, n-1]\cap\mathbb{Z}, n_1+n_2+n_3 > 0\}. \tag{4.108}$$

for $\mathbb{C}^n$. As with the other cases we also could have obtained these two rings by quotienting the $W = 0$ chiral ring by the the two minimal prime ideals over $\mathcal{I}_0$.

4.8.5 Conclusion

We have seen in this section another family of orbifold theories that have a very interesting relationship between transverse space, moduli space, $U(1)$ chiral ring and Fock space of branes. In particular we saw that

1. The moduli space of this theory has two separate branches: one main branch where the space is the same as the space transverse to the D3-branes, and one extra branch which is $\mathbb{C}^n$.

2. The chiral ring of the $U(1)$ theory is equal to the ring of functions on the moduli space, $\mathbb{C}^3/\hat{A}_n \cup \mathbb{C}^n$, rather than the ring of functions on the transverse space, $\mathbb{C}^3/\hat{A}_n$.

3. The $U(1)$ chiral ring has a set of elements that is not equal to the set of single-trace operators in the large N theory.

4. The multi-trace operator generating function gave us the generating function for the Fock space of bosons moving on $\mathbb{C}^3/\hat{A}_n\,[\coprod \mathbb{C}]^{n-1}$.

4.9 Conclusions

In this work, we have described how certain seemingly simple relationships between chiral rings, moduli spaces, generating functions, and transverse geometries get modified in all but the most symmetric examples. While there is generally a main branch of moduli space which is the geometry transverse to the branes, or its symmetric products, there are often extra branches of moduli space. This is a source of subtleties in the generating functions counting chiral ring operators. They also modify the naïve relationship between the generating function of multi-trace operators and the Fock space of bosons moving on the space transverse to the branes.

There are a number of natural questions for future work. Since we have here only considered the $N = 1$ and $N \to \infty$ limits, it would potentially be interesting to understand how our results are modified for finite $N > 1$. The explicit derivation of $Sym^N(X)$ from the chiral ring, where X is the transverse space, is only known in a few cases. From our studies at $N = 1$ and $N \to \infty$, we expect this structure to be present, but with subtle modifications due to the extra branches. A systematic description and derivation of this structure would be fascinating. Restricting attention to the $N = 1$ and $N \to \infty$ cases, is there a simple general mathematical/geometrical formulation (bypassing explicit gauge theory computations), perhaps based on physical ideas around fractional branes, which can start from the chiral ring and moduli space at $N = 1$ and derive the Fock space structure at large N? As we have seen in the examples, the Fock space structure we find from the gauge theory chiral operators always contains a factor which is the Fock space for the main branch corresponding to the geometry transverse to the 3-branes. However, while the extra Fock space factors are correlated with the existence of extra branches, there is no simple rule like the existence of a Fock space at $N \to \infty$ for every branch at $N = 1$. Is there a clear rule which replaces this naive rule? Even if such a rule existed, what would be the geometrical/mathematical meaning behind it?

One of the motivations behind the present work was to ask whether there is a simple general algorithm to deform the large N counting formulae at zero superpotential to arrive at those for non-zero superpotential. The latter have been the main focus of this work. The former admit simple general expressions based on the weighted adjacency matrix of the quiver graph [88, 90]. These expressions are also, somewhat surprisingly from a physical point of view, related to some word-counting problems based on the quiver [90]. We might hope that a deeper geometrical understanding of the role of multiple branches at $N = 1$ and $N \to \infty$ might provide useful hints in finding such an algorithm.

For the case of D3-branes at the tip of a general toric Calabi-Yau cone, all such theories can be reached via Higgsing $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ theories, so it would nice to understand how the generating functions behave under the resulting flows. Similarly, one could hope to understand the relationship of generating functions in theories connected by (relevant) superpotential deformations. Although such deformations are in general complicated, and can lead to vastly different solutions to the F-terms, it might be possible to understand the effect of adding mass terms or other such very simple deformations.

Appendix A

Appendix: Extra Results For $\mathcal{N} = 1$ Theories

A.1 S_ℓ Theories With A T_{N,k_1} And A T_{N,k_2}

In section 3.5.2 we looked at the S_ℓ theory with two $T_{N,k}$'s. Here we summarise the results for the S_ℓ theory with a T_{N,k_1} at one end of the quiver and a T_{N,k_2} at the other end.

As for the S_ℓ theory with two $T_{N,k}$'s we have an R-symmetry which is the same as that given in equation (3.24) and also an additional anomaly-free $U(1)$ symmetry (3.25). In contrast to the S_ℓ theory with two $T_{N,k}$'s it is no longer the case that $\operatorname{Tr}\mathcal{F} = 0$ when ℓ is even and so we must use a-maximization for all ℓ. If we do this we find that the value of α that maximises a is

$$\widehat{\alpha} = \frac{A + \sqrt{B}}{C}, \tag{A.1}$$

where

$$\begin{aligned}
A =& -3N^2\left(k_1 + k_2 + \ell\right) + k_1^3 + k_2^3 - k_1 - k_2, \\
B =& N^4\left(18k_1\ell + 18k_2\ell + 36k_1^2 + 36k_2^2 - 102k_1 + 72k_1k_2 - 102k_2 + 9\ell^2 + 91\right) \\
& + N^2\left(-6k_1^3\ell + 6k_1\ell - 6k_2^3\ell + 6k_2\ell - 24k_1^4 - 24k_2k_1^3 + 10k_1^3 - 24k_2k_1^2 + 40k_1^2\right. \\
& \qquad \left. - 24k_2^3k_1 - 24k_2^2k_1 + 102k_1 - 24k_2^4 + 10k_2^3 + 40k_2^2 + 102k_2 - 160\right) \\
& + 4k_1^6 + 8k_1^5 + 4k_1^4 - 32k_1^3 - 32k_1^2 + 8k_2^3k_1^3 + 8k_2^2k_1^3 + 8k_2^3k_1^2 + 8k_2^2k_1^2 \\
& + 4k_2^6 + 8k_2^5 + 4k_2^4 - 32k_2^3 - 32k_2^2 + 64 \\
C =& \left(-18k_1 - 18k_2 + 42\right)N^2 + 6k_1^3 + 12k_1^2 + 6k_1 + 6k_2^3 + 12k_2^2 + 6k_2 - 48.
\end{aligned}$$

We do not plot this here as the plots are much the same as those in figure 3.8 however we note that it approaches

$$\frac{-3\left(k_1 + k_2 + \ell\right) + \sqrt{18\left(k_1 + k_2\right)\ell + 72k_1k_2 - 102\left(k_1 + k_2\right) + 36\left(k_1^2 + k_2^2\right) + 9\ell^2 + 91}}{6\left(-3k_1 - 3k_2 + 7\right)}$$

at large N. One can verify that this never goes below $-\frac{1}{6}$ and so any gauge-invariant operators that can be constructed satisfy the unitarity bound $R \geq \frac{2}{3}$.

A.2 S_ℓ Theories With Adjoint Matter

In section 3.5.4 we looked at what happens if we take the S_ℓ theory and give a vev to the k-th hypermultiplet. The theory that we get in IR is that represented by the bottom quiver diagram in figure 3.11. It is an S_ℓ theory with an adjoint chiral superfield and extra $Q\Phi\widetilde{Q}$ superpotential terms. For this theory with two $T_{N,k}$'s at each end of the quiver the R-symmetry is $R_{IR} = R_0 + \widehat{\alpha}\mathcal{F}$, where R_0 is given in equation (3.24) with $R_0(\Phi) = 1$, $\mathcal{F}$ is given in equation (3.31) and $\widehat{\alpha}$ is found using a-maximization to be

$$\widehat{\alpha} = \frac{A + \sqrt{B}}{C}, \tag{A.2}$$

where

$$\begin{aligned}
A =& 96N^3 + 72\ell N^2 - 96N, \\
B =& N^6 \left(13824(-1)^l + 23040\right) \\
&+ N^5 \left(20736(-1)^{k+l} + 20736(-1)^k + 13824l - 57600(-1)^l - 57600\right) \\
&+ N^4 \left(-43200(-1)^{k+l} - 43200(-1)^k + 5184l^2 + 44640(-1)^l + 41760\right) \\
&+ N^3 \left(-25344(-1)^{k+l} - 25344(-1)^k - 13824l + 62208(-1)^l + 62208\right) \\
&+ N^2 \left(59904(-1)^{k+l} + 59904(-1)^k - 64512(-1)^l - 82944\right) \\
&+ N \left(9216(-1)^{k+l} + 9216(-1)^k - 18432(-1)^l - 18432\right) \\
&- 18432(-1)^{k+l} - 18432(-1)^k + 18432(-1)^l + 27648, \\
C =& \left(288(-1)^l + 288\right) N^3 + N^2 \left(432(-1)^k - 792(-1)^l - 792\right) \\
&- 576(-1)^k + 576(-1)^l + 576.
\end{aligned}$$

When we do this for the same theory but with a T_{N,k_1} at one end of the quiver and a T_{N,k_2} at the other end, the IR R-symmetry is again given by $R_{IR} = R_0 + \widehat{\alpha}\mathcal{F}$. As the expression for $\widehat{\alpha}$ would take up too much space, we do not include it here.

A.3 Simplification of ST Generating Function for $\mathcal{N} = 4$ SYM With $W \neq 0$

In section 4.3.3 it was said that the generating function for $\mathcal{N} = 4$ SYM with non-zero superpotential could be derived using the Pólya enumeration theorem with $G = S_n$. We derive this now.

The symmetric group has

$$\frac{n!}{\prod_{k=1}^{n}(k!)^{j_k}}\prod_{k=1}^{n}\left(\frac{k!}{k}\right)^{j_k}\prod_{k=1}^{n}\frac{1}{j_k!} \tag{A.1}$$

elements each with j_k cycles of length k for each partition $\{j_k\}$ of n. Thus the cycle index for the symmetric group is

$$Z_{S_n}\left(t_1,...,t_n\right)=\sum_{\{j_k\}\vdash n}\frac{1}{\prod_{k=1}^{n}k^{j_k}\left(j_k!\right)}\prod_{k=1}^{n}t_k^{j_k}. \tag{A.2}$$

The notation $\{j_k\}\vdash n$ means $\{j_k\}$ is a partition of n so the sum is over partitions of n. This means that the generating function for large N $\mathcal{N}=4$ SYM single-trace operators with non-zero superpotential is

$$F_S^{(\infty)}\left(x,y,z\right)=\sum_{n=1}^{\infty}\sum_{\{j_k\}\vdash n}\prod_{k=1}^{n}\frac{(x^k+y^k+z^k)^{j_k}}{k^{j_k}\left(j_k!\right)}. \tag{A.3}$$

To show that this is equal to $(1-x)^{-1}(1-y)^{-1}(1-z)^{-1}$ we start by changing the sum over n and the sum over partitions of n to a infinite number of sums

$$\begin{aligned}
F_S^{(\infty)}\left(x,y,z\right)&=\sum_{n=1}^{\infty}\sum_{\{j_k\}\vdash n}\prod_{k=1}^{n}\frac{1}{j_k!}\left(\frac{x^k+y^k+z^k}{k}\right)^{j_k}\\
&=\left(\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}...\right)\prod_{k=1}^{n}\frac{1}{j_k!}\left(\frac{x^k+y^k+z^k}{k}\right)^{j_k}\\
&=\left(\sum_{j_1=0}^{\infty}\frac{1}{j_1!}\left(\frac{x+y+z}{1}\right)^{j_1}\right)\left(\sum_{j_2=0}^{\infty}\frac{1}{j_2!}\left(\frac{x^2+y^2+z^2}{2}\right)^{j_2}\right)...\\
&=\prod_{k=1}^{\infty}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x^k+y^k+z^k}{k}\right)^{n}\right]=\prod_{k=1}^{\infty}\exp\left(\frac{x^k+y^k+z^k}{k}\right)\\
&=\exp\left(\sum_{k=1}^{\infty}\frac{x^k}{k}\right)\exp\left(\sum_{k=1}^{\infty}\frac{y^k}{k}\right)\exp\left(\sum_{k=1}^{\infty}\frac{z^k}{k}\right)=\frac{1}{1-x}\frac{1}{1-y}\frac{1}{1-z}
\end{aligned} \tag{A.4}$$

A.4 Derivation of $\mathbb{C}^3/\mathbb{Z}_2$ Molien Series

Here we calculate the single-trace operator generating function in the $\mathbb{C}^3/\mathbb{Z}_2$ theory using the methods given in [70]. In [70] it is said that the counting function for a $\mathbb{C}^3/G$ theory is given by the counting function for $\mathbb{C}^3$ polynomials that are invariant under the action of the group G. It is also said that this is a classical problem and that the counting function is given by the Molien series:

$$M\left(t;G\right)=\frac{1}{|G|}\sum_{g\in G}\frac{1}{\det\left(\mathbb{I}-tg\right)}. \tag{B.1}$$

Since $\mathbb{Z}_2$ has two elements whose action on the co-ordinates x, y, z is given by the matrices

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \qquad \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}. \tag{B.2}$$

The Molien series is just

$$M\left(t;\mathbb{Z}_2\right)=\frac{1}{2}\left(\frac{1}{(1-t)^3}+\frac{1}{(1-t)(1+t)^2}\right)=\frac{\left(1+t^2\right)}{(1-t)^3(1+t)^2}. \tag{B.3}$$

This does not match the formula we have given in equation (4.54).

A.5 Derivation of Generating Functions for $\mathbb{C}^3/\mathbb{Z}_n$

First let us consider single-trace operators that have no C operators in them. These are all of the form $\mathrm{Tr}\left(A_{1,2}B_{2,3}A_{3,4}A_{4,5}...A_{n-2,n-1}B_{n-1,n}B_{n,1}\right)$, i.e. A's and B's all over the place. As required by gauge-invariance the number of A's and B's in the trace combined will be ℓn where ℓ is the number of loops we have traced around the quiver by following the arrows of the bifundamentals. The F-term equations allow us to interchange A's and B's freely (e.g. $A_{1,2}B_{2,3}=B_{1,2}A_{2,3}$) so that operators with the same number of A's and the same number of B's are equivalent.

When constructing a gauge-invariant single-trace operator from A's and B's the A and B operators move us from one node to the next and so we need n of them to get back to the node we started at. The C operators move us forward $n-2$ nodes. So for general n the lowest lying single-trace operators have $\#A$'s $+\#B$'s $+\#C$'s $=3$ and are $\mathrm{Tr}\left(A_{1,2}A_{2,3}C_{3,1}\right)$, $\mathrm{Tr}\left(A_{1,2}B_{2,3}C_{3,1}\right)$ and $\mathrm{Tr}\left(B_{1,2}B_{2,3}C_{3,1}\right)$. The F-term relations tells us that any operator with n_1 A's, n_2 B's and n_3 C's is equivalent to any other operator with n_1 A's, n_2 B's and n_3 C's.

We can split the operators up into how many loops they do around the quiver. So, let's take the example of the $\mathbb{C}^3/\mathbb{Z}_5$ theory and for the moment neglect the existence of the B operators. At one loop there are 2 operators:

$$A^5, A^2C, \tag{C.1}$$

At two loops we have 4 operators:

$$A^{10}, A^7C, A^4C^2, AC^3, \tag{C.2}$$

At three loops we have 6 operators:

$$A^{15}, A^{12}C, A^9C^2, A^6C^3, A^3C^4, C^5. \tag{C.3}$$

And so on for higher number of loops.

Each time we have started with ℓn A's and gone from left to right by replacing $n-2=3$ A's with a C. This means that the generating function will be

$$\begin{aligned} F_S^{(\infty)}(a,c) &= \left(1+a^5+a^{10}+a^{15}+\ldots\right)+c\left(a^2+a^7+a^{12}+\ldots\right)+c^2\left(a^4+a^9+\ldots\right) \\ &\quad +c^3\left(a+a^6+\ldots\right)+c^4\left(a^3+a^8+\ldots\right)+c^5\left(1+a^5+a^{10}+\ldots\right)+\ldots \\ &= \left[\sum_{m=0}^{\infty} c^{nm}\right]\Big[\left(1+a^5+a^{10}+a^{15}+\ldots\right)+c\left(a^2+a^7+a^{12}+\ldots\right) \\ &\qquad +c^2\left(a^4+a^9+\ldots\right)+c^3\left(a+a^6+\ldots\right)+c^4\left(a^3+a^8+\ldots\right)\Big] \\ &. \end{aligned} \tag{C.4}$$

Then for general odd n the generating function is

$$F_S^{(\infty)}(a,c) = \left[\sum_{m=0}^{\infty} c^{nm}\right]\left[\sum_{j=0}^{\frac{n-1}{2}} c^j \sum_{\ell=0}^{\infty} a^{n\ell+2j} + c^{\frac{n+1}{2}} \sum_{j=0}^{\frac{n-3}{2}} c^j \sum_{\ell=0}^{\infty} a^{n\ell+2j+1}\right]. \tag{C.5}$$

When we re-introduce the B operators this becomes

$$\boxed{F_S^{(\infty)}(a,b,c) = \left[\sum_{m=0}^{\infty} c^{nm}\right]\left[\sum_{j=0}^{\frac{n-1}{2}}\sum_{\ell=0}^{\infty}\sum_{k=0}^{n\ell+2j} a^k b^{n\ell+2j-k} c^j + c^{\frac{n+1}{2}}\sum_{j=0}^{\frac{n-3}{2}}\sum_{\ell=0}^{\infty}\sum_{k=0}^{n\ell+2j+1} a^k b^{n\ell+2j+1-k} c^j\right]}. \tag{C.6}$$

If we make the replacements $a \to t$, $b \to t$, $c \to t$ in equation (C.6) so that we only have a chemical potential for the R-charge then we get

$$F_S^{(\infty)}(t) = \frac{-t^{2n}-nt^{n+3}-t^{2n+3}-2t^{\frac{3(n+1)}{2}}+2t^{\frac{n+3}{2}}+nt^n+t^3+1}{\left(t^3-1\right)^2\left(t^n-1\right)^2}. \tag{C.7}$$

It can be shown that the result that can be obtained using the methods of [70] for general odd n matches equation (C.7).

We can take the plethystic exponential to get the multi-trace operator generating function:

$$F_M^{(\infty)}(a,b,c) = \prod_{m=0}^{\infty}\left[\prod_{j=0}^{\frac{n-1}{2}}\prod_{\ell=0}^{\infty}\prod_{k=0}^{n\ell+2j}\frac{1}{1-c^{nm+j}a^k b^{n\ell+2j-k}}\right]\left[\prod_{j=0}^{\frac{n-3}{2}}\prod_{\ell=0}^{\infty}\prod_{k=0}^{n\ell+2j+1}\frac{1}{1-c^{nm+\frac{n+1}{2}+j}a^k b^{n\ell+2j-k}}\right]. \tag{C.8}$$

Now let's see what happens for even n. Again let's simplify to the case with only A

and C operators, i.e. no B's. For even n we have to deal with the complication that not all operators with the same number of C's will be equal, e.g. $\mathrm{Tr}(C_{1,n-1}\dots C_{3,1}) \neq \mathrm{Tr}(C_{2,n}\dots C_{4,2})$. Operators with at least one A in the trace will not have this feature. That is to say, all operators with n_1 A's and n_2 C's will be equal to one another when $n_1 > 0$. So our generating function is

$$F_S^{(\infty)}(a,c) = \sum_{m=0}^{\infty}\left(c^{\frac{n}{2}}\right)^m \sum_{\ell=0}^{\infty} a^{n\ell} \sum_{j=0}^{\frac{n}{2}-1} c^j a^{2j} + \sum_{m=1}^{\infty}\left(c^{\frac{n}{2}}\right)^m. \tag{C.9}$$

When we re-introduce the B's this becomes

$$\boxed{F_S^{(\infty)}(a,b,c) = \sum_{m=0}^{\infty} c^{\frac{nm}{2}} \sum_{j=0}^{\frac{n}{2}-1}\sum_{\ell=0}^{\infty}\sum_{k=0}^{n\ell+2j} a^{n\ell+2j-k} b^k c^j + \sum_{m=1}^{\infty} c^{\frac{nm}{2}}}. \tag{C.10}$$

When we make the substitution $a \to t$, $b \to t$, $c \to t$ in equation (C.10) we get

$$F_S^{(\infty)}(t) = \frac{\frac{n\left(1-\left(t^3\right)^{n/2}\right)t^n}{(1-t^3)(1-t^n)^2} + \frac{1-\left(t^3\right)^{n/2}}{(1-t^3)(1-t^n)} + \frac{2\left(-\frac{1}{2}nt^{3n/2}+\left(\frac{n}{2}-1\right)t^{\frac{3n}{2}+3}+t^3\right)}{(1-t^3)^2(1-t^n)}}{1-t^{n/2}} + \frac{t^{n/2}}{1-t^{n/2}}. \tag{C.11}$$

It can be shown that the result that can be obtained using the methods of [70] for general even n is equal to the first term in equation (C.11). The second term accounts for the fact that there are two single-trace operators with $\frac{nm}{2}$ C's for every $m \in \mathbb{Z}_+$.

If we take the plethystic exponential of this then we get the multi-trace operator generating function

$$F_M^{(\infty)}(a,b,c) = \left[\prod_{m=0}^{\infty}\prod_{j=0}^{\frac{n}{2}-1}\prod_{\ell=0}^{\infty}\prod_{k=0}^{n\ell+2j} \frac{1}{1-a^{n\ell+2j-k}b^k c^{j+\frac{nm}{2}}}\right]\left[\prod_{m=1}^{\infty}\frac{1}{1-c^{\frac{nm}{2}}}\right]. \tag{C.12}$$

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